THE GYROSCOPE

THEORY AND APPLICATIONS

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Simple Gyroscope
(Photograph courtesy Sperry Gyroscope Co.)

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PREFACE

The aim of this book is to give a sound, systematic, unmistakably clear and reasonably complete treatment of the mathematical and mechanical aspects of the gyroscope and its more important applications. Since the book is concerned with fundamental principles, constructional details are not entered into except insofar as they may aid in clarifying the principles. Friction has been left out of consideration in the motion of the gyroscope itself, because the construction, mounting and casing of modern gyroscopes are such as to keep friction of all kinds as low as possible. Unavoidable friction is taken care of by the driving motor.

Because I think the mathematical theory of the gyroscope and its applications are best treated with the aid of vectors, I have used vector methods throughout the book. And since a knowledge of vector analysis is not assumed on the part of the reader, the first chapter is devoted to the exposition of the amount of vector analysis needed in the subsequent chapters.

The right-handed system of coordinate axes is used throughout the book, because of its advantages for this work.

A complete and exact mathematical treatment of gyroscopic motion becomes intractable almost at the beginning. Approximations of minor importance must be made in order to obtain tractable and solvable equations. When making simplifying approximations, I have pointed out their nature and in some cases have shown by numerical examples that the errors thus introduced were of no consequence.

In the preparation of this book I have consulted the works of many previous writers, the most important of which are listed in the Bibliography at the end of the book.

It is a pleasure to record my thanks and obligations to the Sperry Gyroscope Company, Great Neck, New York, for their unstinted cooperation in furnishing information and photographs relating to various gyroscopic instruments and applications. I also wish to record my thanks to the following other manufacturers for

furnishing information and photographs: The Arma Corporation, Garden City, New York; The Minneapolis-Honeywell Regulator Company, Minneapolis, Minnesota; and The National Engineering Company, Chicago, Illinois. Finally, I wish to thank Mr. Donald Trumpy, of the yacht-building firm of John Trumpy and Sons, Annapolis, Maryland, for gyroscopic data concerning a 29,000-mile cruise which he took on a yacht equipped with a Sperry ship stabilizer.

J. B. Scarborough

August, 1957

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PART I

THEORY OF THE GYROSCOPE

CHAPTER I

Some Necessary Vector Analysis

The theory, behavior and applications of the gyroscope can be explained best by means of vectors. We therefore devote the present chapter to the exposition of the needed amount of vector analysis.

1. Scalar and vector quantities

The quantities which occur in physics are of two kinds: those which have magnitude only and those which have both magnitude and direction. The former are called *scalar* quantities and the latter are called *vector* quantities. Examples of scalar quantities are density, temperature and electric potential. Familiar examples of vector quantities are force, velocity and acceleration. Vectors may also be used to denote position, in which case they are called position vectors.

Vector quantities are represented geometrically by segments of straight lines, the line segments carrying arrow heads at one end to indicate the sense of the vector. The magnitude and direction of the directed quantity are indicated by the length and orientation of the vector.

In Fig. 1 is shown a vector $\overrightarrow{AB}$, which is also denoted by the single letter $\mathbf{r}$. The projections of $\overrightarrow{AB}$ on the coordinate axes are the vectors $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ as shown.

A vector is designated in print by a single boldface letter, as $\mathbf{r}$ in Fig. 1; or by the letters designating its end-points, with an arrow written over the two letters, as $\overrightarrow{AB}$ in Fig. 1.

Vectors may be multiplied or divided at will by any numbers or scalars, the results always being vectors. Thus $3\mathbf{r}$, $\overrightarrow{AB}/l$, etc., are vectors having the same direction and sense as the original vectors. However, the multiplication or division of a vector by a negative number always reverses the sense of the vector.

Two vectors are *equal* when they have the same magnitude, direction and sense.

I-2

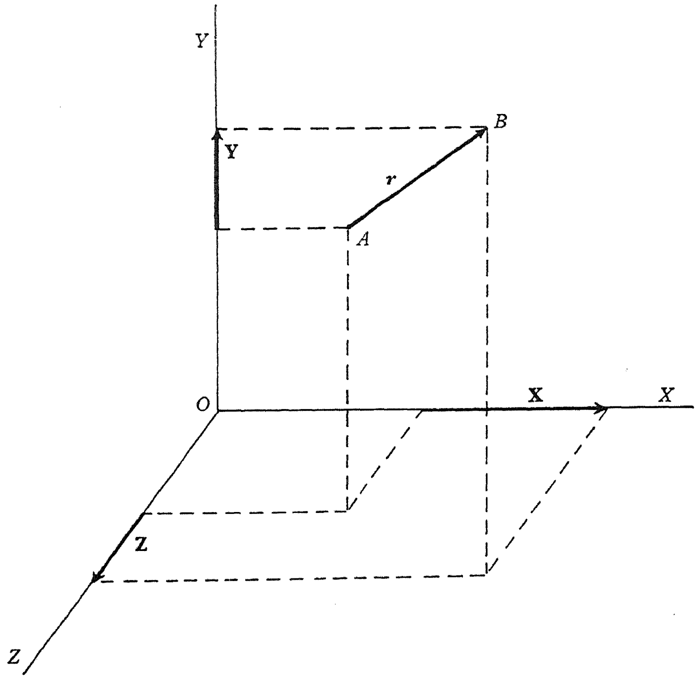


Fig. 1

2. Geometric addition and subtraction of vectors

To find the geometric sum of two vectors, place the initial point of the second vector at the terminal point of the first and then draw a line from the initial point of the first vector to the terminal point of the second. The vector thus drawn is the geometric sum of the given vectors. In Fig. 2, for example, $\overrightarrow{AC}$ is the geometric or vector sum of $\overrightarrow{AB}$ and $\overrightarrow{BC}$. Such addition is denoted by either of the equations

$$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}, \tag{2·1}$$

$$\mathbf{P}+\mathbf{Q}=\mathbf{R}. \tag{2·2}$$

The geometric difference between two vectors is found by changing the sign of the subtrahend vector and then adding it to the other vector. Thus

$$\mathbf{P}-\mathbf{Q}=\mathbf{P}+(-\mathbf{Q}). \tag{2·3}$$

Reference to Fig. 2 will show that vectors are added and subtracted by the parallelogram law, the sum being given by one

diagonal of the parallelogram and the difference by the other diagonal.

The geometric sum of any number of vectors is found by the same procedure as in the case of two vectors; that is, the initial point of the second vector is placed at the terminal point of the first, the

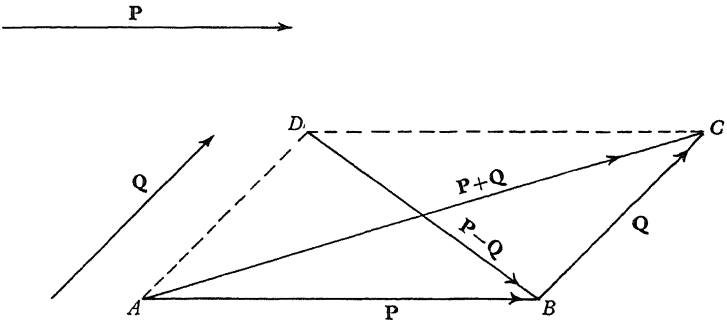


Fig. 2

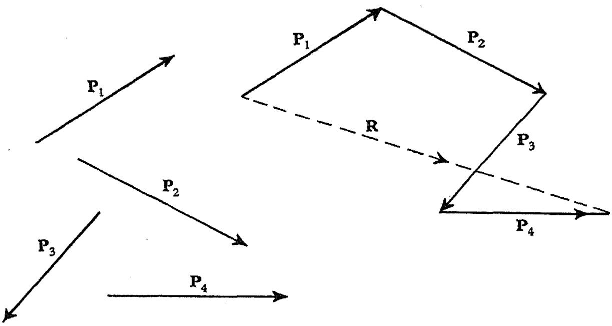


Fig. 3

initial point of the third is placed at the terminal point of the second, etc. The geometric sum of all the vectors is the vector drawn from the initial point of the first to the terminal point of the last, as indicated in Fig. 3.

3. Analytical addition of vectors

The sum or resultant of several vectors is found analytically by first resolving the vectors into rectangular components along coordinate axes, finding the algebraic sums of these components

(projections on the coordinate axes), and then finding the resultant sum in magnitude and direction by the methods of analytic geometry. Thus, if the vectors $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \ldots, \mathbf{P}_n$ make the angles $\alpha_1, \beta_1, \gamma_1; \alpha_2, \beta_2, \gamma_2; \ldots \alpha_n, \beta_n, \gamma_n$ with the coordinate axes, we have

$$\left.\begin{aligned}\Sigma\mathbf{X} &= \mathbf{P}_1\cos\alpha_1 + \mathbf{P}_2\cos\alpha_2 + \ldots + \mathbf{P}_n\cos\alpha_n,\\ \Sigma\mathbf{Y} &= \mathbf{P}_1\cos\beta_1 + \mathbf{P}_2\cos\beta_2 + \ldots + \mathbf{P}_n\cos\beta_n,\\ \Sigma\mathbf{Z} &= \mathbf{P}_1\cos\gamma_1 + \mathbf{P}_2\cos\gamma_2 + \ldots + \mathbf{P}_n\cos\gamma_n.\end{aligned}\right\} \tag{3·1}$$

Then the magnitude of the resultant is

$$R = \sqrt{\{(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2\}}, \tag{3·2}$$

and its direction cosines are

$$\alpha = \cos^{-1}\frac{\Sigma X}{R}, \quad \beta = \cos^{-1}\frac{\Sigma Y}{R}, \quad \gamma = \cos^{-1}\frac{\Sigma Z}{R}. \tag{3·3}$$

4. The unit coordinate vectors i, j, k

For many purposes it is convenient and advantageous to express vectors in terms of *unit vectors* along the coordinate axes. Let $\mathbf{i}$ denote a vector of unit length along the x-axis, let $\mathbf{j}$ denote a unit vector along the y-axis, and let $\mathbf{k}$ denote a unit vector along the z-axis, as shown in Fig. 4. Let $\mathbf{r}$ be a position vector drawn from the origin to the point $P(x, y, z)$. Then the projections of $\mathbf{r}$ on the coordinate axes are the vectors $x\mathbf{i}$, $y\mathbf{j}$ and $z\mathbf{k}$; and since a vector is equal to the geometric sum of its projections (Art. 2, Fig. 3), we may write

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}. \tag{4·1}$$

If r_x, r_y, r_z or r_1, r_2, r_3 are the magnitudes of the projections of $\mathbf{r}$ on the coordinate axes, then $\mathbf{r}$ may be written in either of the forms

$$\mathbf{r} = r_x\mathbf{i} + r_y\mathbf{j} + r_z\mathbf{k}, \tag{4·2}$$

$$\mathbf{r} = r_1\mathbf{i} + r_2\mathbf{j} + r_3\mathbf{k}. \tag{4·3}$$

The vector $\mathbf{r}$ is sometimes written in the form $[\mathbf{r}_x, \mathbf{r}_y, \mathbf{r}_z]$.

Let us now consider two vectors $\mathbf{p}$ and $\mathbf{q}$. They may be expressed in the forms

$$\mathbf{p} = p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k},$$

$$\mathbf{q} = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k},$$

where p_1, p_2, p_3 and q_1, q_2, q_3 are the magnitudes of the rectangular components of **p** and **q**, respectively. Adding **p** and **q**, we get

$$\mathbf{p}+\mathbf{q}=(p_1+q_1)\,\mathbf{i}+(p_2+q_2)\,\mathbf{j}+(p_3+q_3)\,\mathbf{k}. \tag{4·4}$$

The geometric sum of any number of vectors may likewise be expressed in terms of the unit coordinate vectors.

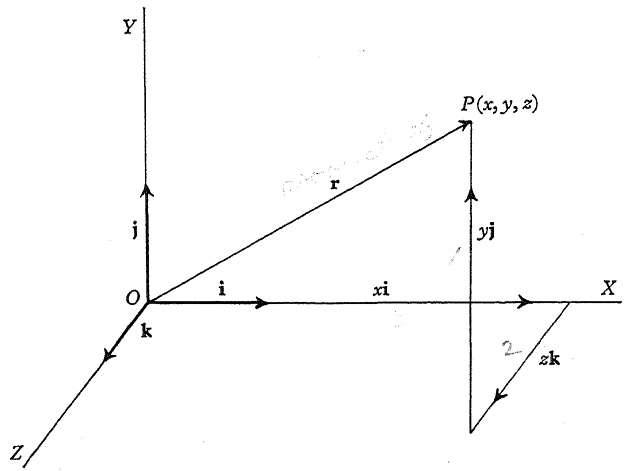


Fig. 4

5. The scalar or dot product of two vectors

We have already pointed out that a vector may be multiplied or divided by abstract numbers and scalars at pleasure, the resulting product or quotient always being a vector along the line of the original vector. But the product of one vector by another is an entirely different matter and must be explicitly defined before proceeding further.

There are two distinct kinds of vector multiplication: scalar multiplication and vector multiplication. The *scalar* or *dot product* of two vectors **a** and **b** is indicated and defined as follows:

$$\mathbf{a}\cdot\mathbf{b}=ab\cos\theta, \tag{5·1}$$

where θ is the angle between **a** and **b** (Fig. 5). The dot product is thus a scalar.

Since $a\cos\theta$ is the length of the projection of **a** on **b**, and $b\cos\theta$

is the length of the projection of **b** on **a**, *the scalar product of two vectors is equal to the length of either vector multiplied by the length of the projection of the other upon it.*

If the vectors **a** and **b** are perpendicular to each other, then $\theta = 90°$, $\cos\theta = 0$, and $\mathbf{a}\cdot\mathbf{b} = 0$. Hence *the condition for perpendicularity of two vectors is that their scalar product be zero.*

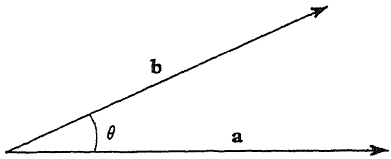


Fig. 5

If **a** and **b** are parallel, then $\theta = 0$, $\cos\theta = 1$ and $\mathbf{a}\cdot\mathbf{b} = ab$. In particular,

$$\mathbf{a}\cdot\mathbf{a} = a^2. \tag{5·2}$$

From (5·1) it is plain that

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}. \tag{5·3}$$

The order of the factors in the dot product is thus immaterial.

The following relations among the dot products of the unit coordinate vectors are of great importance and should always be kept in mind:

$$\left.\begin{aligned} \mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} &= \mathbf{k}\cdot\mathbf{k} = 1, \\ \mathbf{i}\cdot\mathbf{j} = \mathbf{i}\cdot\mathbf{k} &= \mathbf{j}\cdot\mathbf{k} = 0. \end{aligned}\right\} \tag{5·4}$$

We shall now show that the distributive law of multiplication holds for the scalar product of two vectors. Consider a vector **a** and two other vectors **b** and **c**. We are to show that

$$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}. \tag{5·5}$$

Proof. In Fig. 6 consider the vector $\overrightarrow{OA} = \mathbf{a}$ and the projections of the other vectors upon it. We have

$$(a) \quad \mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = (\overline{OA})\,(\overline{FH}),$$

$$(b) \quad \mathbf{a}\cdot\mathbf{b} = (\overline{OA})\,(\overline{FG}),$$

$$(c) \quad \mathbf{a}\cdot\mathbf{c} = (\overline{OA})\,(\overline{GH}).$$

Adding (*b*) and (*c*), we get

$$(d) \quad \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}=(\overline{OA})(\overline{FG}+\overline{GH})=(\overline{OA})(\overline{FH}).$$

Since the right-hand members of (*a*) and (*d*) are the same, we have

$$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}. \qquad \text{Q.E.D.} \tag{5·5}$$

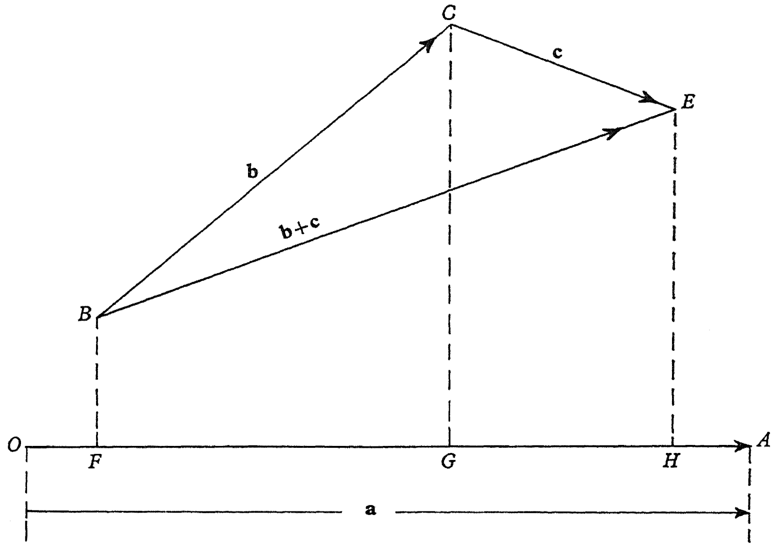


Fig. 6

This result and the relations among the unit coordinate vectors enable us to write the product $\mathbf{a}\cdot\mathbf{b}$ in terms of the rectangular components of $\mathbf{a}$ and $\mathbf{b}$. We have

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b}&=(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\cdot(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})\\&=a_1b_1\mathbf{i}\cdot\mathbf{i}+a_2b_2\mathbf{j}\cdot\mathbf{j}+a_3b_3\mathbf{k}\cdot\mathbf{k}\\&=a_1b_1+a_2b_2+a_3b_3 \quad \text{by (5·4)},\end{aligned}$$

all cross-products such as $a_1b_2\mathbf{i}\cdot\mathbf{j}$ being zero. We thus have the important relation

$$\mathbf{a}\cdot\mathbf{b}=a_1b_1+a_2b_2+a_3b_3. \tag{5·6}$$

6. The vector or cross product of two vectors

The vector product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is indicated and defined as follows:

$$\mathbf{a}\times\mathbf{b}=(ab\sin\theta)\,\mathbf{n}, \tag{6·1}$$

where $\mathbf{n}$ is a *unit vector* perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$, and θ is the angle through which $\mathbf{a}$ must be rotated to make it coincide in direction with $\mathbf{b}$ (Fig. 7). The vector product of two vectors is thus a *vector* perpendicular to the plane of the given vectors and of magnitude equal to the area of the parallelogram having the two given vectors as adjacent sides.

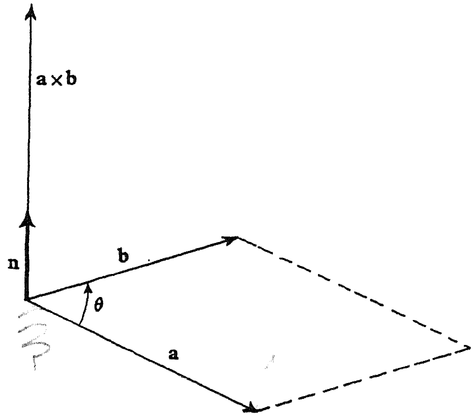


Fig. 7

The sense (indicated by the arrowhead) of the vector $\mathbf{a} \times \mathbf{b}$ is determined by the following rule, called the *right-handed-screw rule*:

The product vector $\mathbf{a} \times \mathbf{b}$ must be drawn in the direction a right-handed screw (corkscrew, for example) would advance if it were rotated in the direction that $\mathbf{a}$ would have to be rotated to make it coincide in direction with $\mathbf{b}$. Thus, a glance at Fig. 7 shows that a right-handed screw would advance upward when rotated in the direction required to bring $\mathbf{a}$ into coincidence with $\mathbf{b}$.

Let us now consider the product $\mathbf{b} \times \mathbf{a}$. In this case a right-handed screw would advance downward if rotated in the direction required to bring $\mathbf{b}$ into coincidence with $\mathbf{a}$. Furthermore, the angle θ in this case would be negative; and since $\sin(-\theta) = -\sin\theta$, we have

$$\mathbf{b} \times \mathbf{a} = [ba \sin(-\theta)]\mathbf{n} = -(ba \sin\theta)\mathbf{n} = -(\mathbf{a} \times \mathbf{b}), \qquad (6\cdot2)$$

thus agreeing with the corkscrew rule.

Equation (6·2) shows that the multiplication is not commutative in finding the vector product of two vectors. The *order of the factors* must be observed. If two factors are interchanged, the sign of the product must be changed.

From (6·1) it is plain that if $\mathbf{a}\times\mathbf{b}=0$, $\mathbf{a}$ and $\mathbf{b}$ are parallel. In particular,

$$\mathbf{a}\times\mathbf{a}=0. \tag{6·3}$$

Since the unit coordinate vectors are mutually perpendicular to one another ($\theta=90°$), we get from (6·1), (6·2) and (6·3), the following relations:

$$\left.\begin{aligned} \mathbf{i}\times\mathbf{j}&=\mathbf{k}=-\mathbf{j}\times\mathbf{i},\\ \mathbf{j}\times\mathbf{k}&=\mathbf{i}=-\mathbf{k}\times\mathbf{j},\\ \mathbf{k}\times\mathbf{i}&=\mathbf{j}=-\mathbf{i}\times\mathbf{k},\\ \mathbf{i}\times\mathbf{i}&=\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}=0.\end{aligned}\right\} \tag{6·4}$$

We now proceed to find the rectangular components of the product vector $\mathbf{a}\times\mathbf{b}$. Let us denote this vector by $\mathbf{c}$, and let the magnitudes of its rectangular components be c_1, c_2, c_3. Then, since $\mathbf{c}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, we have:

$$\mathbf{a}\cdot\mathbf{c}=a_1c_1+a_2c_2+a_3c_3=0,$$

$$\mathbf{b}\cdot\mathbf{c}=b_1c_1+b_2c_2+b_3c_3=0.$$

Solving these equations for c_1 and c_2 in terms of c_3, we get

$$c_1=\frac{c_3(a_2b_3-a_3b_2)}{a_1b_2-a_2b_1},\quad c_2=\frac{c_3(a_3b_1-a_1b_3)}{a_1b_2-a_2b_1}. \tag{6.5}$$

Since $c^2=c_1^2+c_2^2+c_3^2$, we have

$$\begin{aligned} c^2&=\frac{c_3^2(a_2b_3-a_3b_2)^2+c_3^2(a_3b_1-a_1b_3)^2}{(a_1b_2-a_2b_1)^2}+c_3^2\\ &=\frac{c_3^2}{(a_1b_2-a_2b_1)^2}[a_1^2b_2^2+a_1^2b_3^2+a_2^2b_1^2+a_2^2b_3^2+a_3^2b_1^2+a_3^2b_2^2\\ &\qquad -2a_1a_2b_1b_2-2a_1a_3b_1b_3-2a_2a_3b_2b_3].\end{aligned}$$

Now adding and subtracting $a_1^2b_1^2+a_2^2b_2^2+a_3^2b_3^2$ within the brackets of the right-hand member, we obtain

$$\begin{aligned} c^2=\frac{c_3^2}{(a_1b_2-a_2b_1)^2}&[a_1^2b_1^2+a_1^2b_2^2+a_1^2b_3^2+a_2^2b_1^2+a_2^2b_2^2\\ &+a_2^2b_3^2+a_3^2b_1^2+a_3^2b_2^2+a_3^2b_3^2-(a_1^2b_1^2+a_2^2b_2^2+a_3^2b_3^2\\ &\qquad +2a_1a_2b_1b_2+2a_1a_3b_1b_3+2a_2a_3b_2b_3)],\end{aligned}$$

which can be written in the form

$$c^2=\frac{c_3^2}{(a_1b_2-a_2b_1)^2}[(a_1^2+a_2^2+a_3^2)\,(b_1^2+b_2^2+b_3^2)-(a_1b_1+a_2b_2+a_3b_3)^2],$$

or
$$c^2 = \frac{c_3^2}{(a_1 b_2 - a_2 b_1)^2}\,[a^2b^2 - (\mathbf{a}\cdot\mathbf{b})^2] \quad \text{by (5·6),}$$
$$= \frac{c_3^2}{(a_1 b_2 - a_2 b_1)^2}\,[a^2b^2 - a^2b^2\cos^2\theta] \quad \text{by (5·1).}$$

Hence
$$c^2 = \frac{c_3^2 a^2 b^2 \sin^2\theta}{(a_1 b_2 - a_2 b_1)^2},$$

or
$$c = \pm\frac{c_3 ab\sin\theta}{a_1 b_2 - a_2 b_1}. \tag{6·6}$$

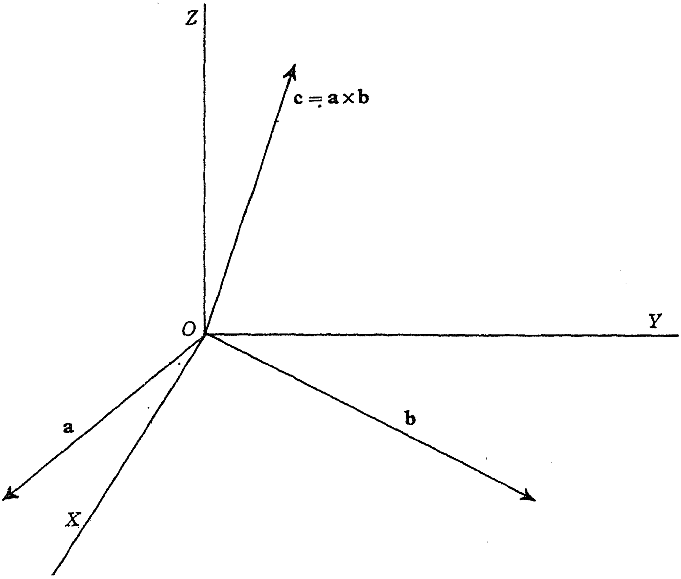


Fig. 8

Now since $\mathbf{c} = c\mathbf{n} = \mathbf{a}\times\mathbf{b} = (ab\sin\theta)\,\mathbf{n}$, we get $c = ab\sin\theta$. Substituting this in (6·6), we get

$$\frac{c_3}{a_1 b_2 - a_2 b_1} = \pm 1. \tag{6·7}$$

To determine which of the two signs on the right should be used, we utilize the fact that all the above relations hold for any values and positions of **a** and **b**. Hence we may take **a** and **b** as unit vectors along the positive x- and y-axes, respectively. Then **c** becomes a

unit vector along the positive z-axis. Hence the rectangular components of **a**, **b** and **c** in this case are

$$a_1=1,\ a_2=0,\ a_3=0; \qquad b_1=0,\ b_2=1,\ b_3=0;$$
$$c_1=0,\ c_2=0,\ c_3=1.$$

Substituting these values in the left-hand member of (6·7), we find that

$$\frac{c_3}{a_1b_2-a_2b_1}=1.$$

Hence $\qquad c_3=a_1b_2-a_2b_1,$

and then (6·5) gives $\qquad c_1=a_2b_3-a_3b_2,$

$$c_2=a_3b_1-a_1b_3.$$

The product vector $\mathbf{a}\times\mathbf{b}$ is therefore

$$\begin{aligned}\mathbf{a}\times\mathbf{b}&=\mathbf{c}=c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}\\ &=(a_2b_3-a_3b_2)\,\mathbf{i}+(a_3b_1-a_1b_3)\,\mathbf{j}+(a_1b_2-a_2b_1)\,\mathbf{k}.\end{aligned}$$

This can be written more compactly in determinant form

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}. \tag{6·8}$$

We are now in a position to prove that the distributive law of multiplication applies to vector products; that is, that

$$\mathbf{a}\times(\mathbf{b}+\mathbf{c})=(\mathbf{a}\times\mathbf{b})+(\mathbf{a}\times\mathbf{c}). \tag{6·9}$$

To prove this extremely important fact we utilize the addition theorem of determinants, namely,

$$\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1+d_1 & c_2+d_2 & c_3+d_3\end{vmatrix}=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}+\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ d_1 & d_2 & d_3\end{vmatrix}.$$

Since the rectangular components of $\mathbf{b}+\mathbf{c}$ are b_1+c_1, b_2+c_2, b_3+c_3, we have, from (6·8),

$$\begin{aligned}\mathbf{a}\times(\mathbf{b}+\mathbf{c})&=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1+c_1 & b_2+c_2 & b_3+c_3\end{vmatrix}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}+\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ c_1 & c_2 & c_3\end{vmatrix}\\ &=(\mathbf{a}\times\mathbf{b})+(\mathbf{a}\times\mathbf{c}) \quad \text{by (6·8).}\end{aligned}$$

It is worth while to point out here that *terms may be transposed from one side of a vector equation to the other*, but that *cancellation of a vector out of the two sides of the equation is not permissible.* For example, suppose we have the vector equation

$$\mathbf{a}\cdot\mathbf{c}=\mathbf{b}\cdot\mathbf{c}.$$

We might be tempted to divide throughout by $\mathbf{c}$ and get $\mathbf{a}=\mathbf{b}$. But we get another solution by transposing $\mathbf{b}\,.\,\mathbf{c}$ to the left-hand side and factoring out $\mathbf{c}$, as we are permitted to do because the scalar

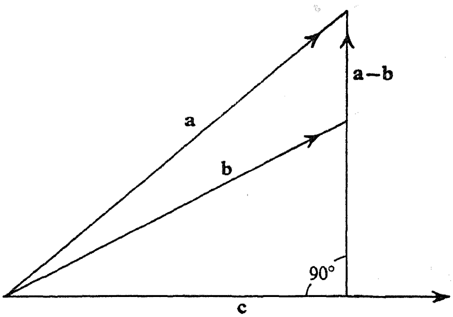


Fig. 9

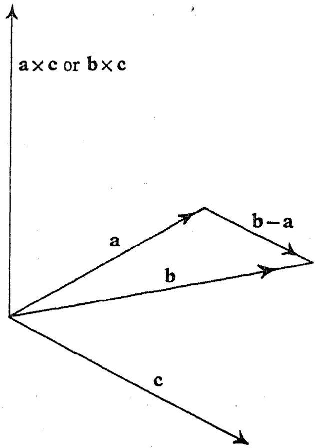


Fig. 10

product obeys the distributive law of multiplication. Then the equation becomes

$$(\mathbf{a}-\mathbf{b})\cdot\mathbf{c}=0.$$

This equation is satisfied if $\mathbf{a}=\mathbf{b}$, but it is also satisfied if $\mathbf{a}\neq\mathbf{b}$ and $\mathbf{c}$ is perpendicular to the vector $\mathbf{a}-\mathbf{b}$. The situation in the latter case is represented by Fig. 9.

In like manner, suppose we are given the vector equation

$$\mathbf{a}\times\mathbf{c}=\mathbf{b}\times\mathbf{c}.$$

The left-hand side of this equation is a vector which is perpendicular to both $\mathbf{a}$ and $\mathbf{c}$, and the right-hand member is a vector which is perpendicular to both $\mathbf{b}$ and $\mathbf{c}$. Since the vectors $\mathbf{a}\times\mathbf{c}$ and $\mathbf{b}\times\mathbf{c}$ are equal, the planes to which they are perpendicular must coincide

and therefore the three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ all lie in the same plane. Let us now transpose both members to the same side of the equation and factor out $\mathbf{c}$ (permissible because the vector product obeys the distributive law). Then we have either

$$(\mathbf{a}-\mathbf{b})\times\mathbf{c}=0 \quad \text{or} \quad (\mathbf{b}-\mathbf{a})\times\mathbf{c}=0.$$

These equations are satisfied if $\mathbf{a}=\mathbf{b}$, but they are satisfied equally well if $\mathbf{a}\neq\mathbf{b}$ and $\mathbf{c}$ is parallel to the vector $\mathbf{a}-\mathbf{b}$. Fig. 10 represents the situation in this latter case.

7. The vector triple product $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$

This vector product is a vector lying in the plane of $\mathbf{b}$ and $\mathbf{c}$, and is of great importance in the study of the gyroscope. Fig. 11 represents the relative positions of the vectors involved in this triple product. The fact that the triple-product vector lies in the plane of $\mathbf{b}$ and $\mathbf{c}$ can readily be seen when it is remembered that the vector $\mathbf{b}\times\mathbf{c}$ is perpendicular to each of the three vectors $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ at the point O. Hence all three of these vectors lie in the plane to which $\mathbf{b}\times\mathbf{c}$ is perpendicular at O.

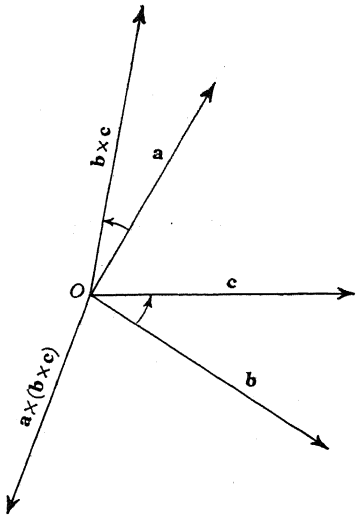


Fig. 11

It is to be noted that

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\neq(\mathbf{a}\times\mathbf{b})\times\mathbf{c};$$

for the vector on the left-hand side lies in the plane of $\mathbf{b}$ and $\mathbf{c}$, whereas the vector on the right-hand side lies in the plane of $\mathbf{a}$ and $\mathbf{b}$; and these planes cannot be the same.* This fact shows that the vector triple product does not obey the associative law of multiplication.

The vector triple product can be expressed as the difference of two vectors, as we shall now show. In Art. 6 the magnitudes of the rectangular components of the vector $\mathbf{b}\times\mathbf{c}$ were found to be

* Since $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ represent any three vectors in space, they are in general noncoplanar.

$b_2c_3-b_3c_2$, $b_3c_1-b_1c_3$ and $b_1c_2-b_2c_1$. Hence by (6·8), we may write the triple product in the form

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_2c_3-b_3c_2 & b_3c_1-b_1c_3 & b_1c_2-b_2c_1 \end{vmatrix}.$$

Expanding this determinant and rearranging the terms slightly, we get

$$\begin{aligned}\mathbf{a}\times(\mathbf{b}\times\mathbf{c})&=b_1(a_2c_2+a_3c_3)\,\mathbf{i}+b_2(a_1c_1+a_3c_3)\,\mathbf{j}\\ &\quad+b_3(a_1c_1+a_2c_2)\,\mathbf{k}-c_1(a_2b_2+a_3b_3)\,\mathbf{i}\\ &\quad-c_2(a_1b_1+a_3b_3)\,\mathbf{j}-c_3(a_1b_1+a_2b_2)\,\mathbf{k}.\end{aligned}$$

The form of the parenthetical expressions suggests that a third product term be added to each of the positive components and subtracted from the corresponding negative component. We therefore add $b_1a_1c_1\mathbf{i}$, $b_2a_2c_2\mathbf{j}$, $b_3a_3c_3\mathbf{k}$ to the positive components and subtract these terms from the corresponding negative components. We then have

$$\begin{aligned}\mathbf{a}\times(\mathbf{b}\times\mathbf{c})&=b_1(a_1c_1+a_2c_2+a_3c_3)\,\mathbf{i}+b_2(a_1c_1+a_2c_2+a_3c_3)\,\mathbf{j}\\ &\quad+b_3(a_1c_1+a_2c_2+a_3c_3)\,\mathbf{k}-c_1(a_1b_1+a_2b_2+a_3b_3)\,\mathbf{i}\\ &\quad-c_2(a_1b_1+a_2b_2+a_3b_3)\,\mathbf{j}-c_3(a_1b_1+a_2b_2+a_3b_3)\,\mathbf{k}.\end{aligned}$$

Now reducing the parenthetical expressions by means of formula (5·6), we get

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=[b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}]\,(\mathbf{a}\cdot\mathbf{c})-[c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}]\,(\mathbf{a}\cdot\mathbf{b}),$$

or, finally, $$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}. \tag{7·1}$$

This is one of the most important formulas of vector analysis. Since the dot products in the parentheses are scalars, the right-hand member of (7·1) is the difference of two vectors.

8. Differentiation of vectors

If a vector $\mathbf{r}$ is a function of a variable parameter t (time, for example), then an increment Δt in t will cause a change $\Delta\mathbf{r}$ in $\mathbf{r}$. The new value of the vector is therefore $\mathbf{r}+\Delta\mathbf{r}$, as indicated in Fig. 12. Then the derivative of $\mathbf{r}$ with respect to t is defined to be

$$\frac{d\mathbf{r}}{dt}=\lim_{\Delta t\to 0}\frac{\Delta\mathbf{r}}{\Delta t}.$$

If $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, then

$$\frac{d\mathbf{r}}{dt}=\mathbf{i}\frac{dx}{dt}+\mathbf{j}\frac{dy}{dt}+\mathbf{k}\frac{dz}{dt}, \tag{8·1}$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ have been treated as constants in the differentiation. More generally,

$$\frac{d^n\mathbf{r}}{dt^n}=\mathbf{i}\frac{d^nx}{dt^n}+\mathbf{j}\frac{d^ny}{dt^n}+\mathbf{k}\frac{d^nz}{dt^n}. \tag{8·2}$$

The derivative of a vector is thus a vector.

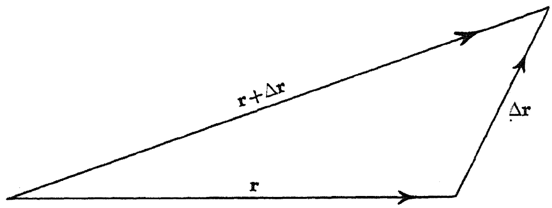


Fig. 12

To find the derivative of the scalar product $\mathbf{a}\cdot\mathbf{b}$ we write

$$P=\mathbf{a}\cdot\mathbf{b}.$$

Then

$$P+\Delta P=(\mathbf{a}+\Delta\mathbf{a})\cdot(\mathbf{b}+\Delta\mathbf{b})=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\Delta\mathbf{b}+\Delta\mathbf{a}\cdot\mathbf{b}+\Delta\mathbf{a}\cdot\Delta\mathbf{b}.$$

$$\therefore\quad \Delta P=\mathbf{a}\cdot\Delta\mathbf{b}+\Delta\mathbf{a}\cdot\mathbf{b}+\Delta\mathbf{a}\cdot\Delta\mathbf{b},$$

and

$$\frac{\Delta P}{\Delta t}=\mathbf{a}\cdot\frac{\Delta\mathbf{b}}{\Delta t}+\mathbf{b}\cdot\frac{\Delta\mathbf{a}}{\Delta t}+\Delta\mathbf{a}\cdot\frac{\Delta\mathbf{b}}{\Delta t}.$$

$$\therefore\quad \frac{dP}{dt}=\mathbf{a}\cdot\frac{d\mathbf{b}}{dt}+\mathbf{b}\cdot\frac{d\mathbf{a}}{dt},$$

or

$$\frac{d(\mathbf{a}\cdot\mathbf{b})}{dt}=\mathbf{a}\cdot\frac{d\mathbf{b}}{dt}+\mathbf{b}\cdot\frac{d\mathbf{a}}{dt}. \tag{8·3}$$

The derivative of the vector product $\mathbf{a}\times\mathbf{b}$ is found by a similar procedure. We write

$$\mathbf{V}=\mathbf{a}\times\mathbf{b}.$$

$$\mathbf{V}+\Delta\mathbf{V}=(\mathbf{a}+\Delta\mathbf{a})\times(\mathbf{b}+\Delta\mathbf{b})=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\Delta\mathbf{b}+\Delta\mathbf{a}\times\mathbf{b}+\Delta\mathbf{a}\times\Delta\mathbf{b}.$$

$$\therefore\quad \Delta\mathbf{V}=\mathbf{a}\times\Delta\mathbf{b}+\Delta\mathbf{a}\times\mathbf{b}+\Delta\mathbf{a}\times\Delta\mathbf{b},$$

and

$$\frac{\Delta\mathbf{V}}{\Delta t}=\mathbf{a}\times\frac{\Delta\mathbf{b}}{\Delta t}+\frac{\Delta\mathbf{a}}{\Delta t}\times\mathbf{b}+\Delta\mathbf{a}\times\frac{\Delta\mathbf{b}}{\Delta t}.$$

Hence
$$\frac{d\mathbf{V}}{dt}=\mathbf{a}\times\frac{d\mathbf{b}}{dt}+\frac{d\mathbf{a}}{dt}\times\mathbf{b},$$

or
$$\frac{d(\mathbf{a}\times\mathbf{b})}{dt}=\mathbf{a}\times\frac{d\mathbf{b}}{dt}+\frac{d\mathbf{a}}{dt}\times\mathbf{b} \tag{8·4}$$
$$=\mathbf{a}\times\frac{d\mathbf{b}}{dt}-\mathbf{b}\times\frac{d\mathbf{a}}{dt}.$$

Note that when the order of the factors is changed in a vector product the sign before the product must also be changed.

If a vector is constant in magnitude, its derivative is not necessarily zero; for the vector is still free to change in direction. Suppose, for example, that

$$r=\text{a constant}.$$

Then
$$r^2=\text{a constant}.$$

That is,
$$\mathbf{r}\cdot\mathbf{r}=\text{a constant}.$$

Differentiating this, we have

$$\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}+\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}=0,$$

or
$$\mathbf{r}\cdot d\mathbf{r}=0.$$

This shows that $d\mathbf{r}$ is perpendicular to $\mathbf{r}$.

CHAPTER II

Some Fundamental Principles of Mechanics

This chapter deals with certain topics in mechanics which are fundamental for the study of the gyroscope.

9. Velocity and momentum

Assume that a particle of mass m is moving along a curve in three-dimensional space, and let $\mathbf{r}$ denote a position vector drawn from a point O to the particle m. Then $\mathbf{r}$ is evidently a function of the time, or $\mathbf{r}=\mathbf{r}(t)$. Let $\mathbf{v}$ denote the velocity of the particle along its path. Then

$$\mathbf{v}=\frac{d\mathbf{r}}{dt}=\dot{\mathbf{r}}.$$

The *momentum* of the particle is $m\mathbf{v}=m\dot{\mathbf{r}}$ and is thus a vector quantity having the same direction and sense as $\mathbf{v}$.

10. The fundamental equation of dynamics

The most fundamental and important principle of dynamics is Newton's second law of motion. It states that

$$\mathbf{F}=\frac{d}{dt}(m\mathbf{v})=\frac{d}{dt}(m\dot{\mathbf{r}}), \tag{10·1}$$

where $\mathbf{F}$ is the resultant of all the forces acting on the particle. If m is constant, as is usually the case, the above equation becomes

$$\mathbf{F}=m\frac{d\mathbf{v}}{dt}=m\frac{d\dot{\mathbf{r}}}{dt}=m\ddot{\mathbf{r}}=m\mathbf{a}, \tag{10·2}$$

where $\mathbf{a}$ denotes the acceleration imparted to the particle by the force $\mathbf{F}$.

Since $\qquad \mathbf{F}=F_x\mathbf{i}+F_y\mathbf{j}+F_z\mathbf{k}=X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k},$

and $\qquad \ddot{\mathbf{r}}=\ddot{x}\mathbf{i}+\ddot{y}\mathbf{j}+\ddot{z}\mathbf{k},$

equation (10·2) is the vector equivalent of the three Cartesian equations

$$\left.\begin{aligned} X &= m\ddot{x}, \\ Y &= m\ddot{y}, \\ Z &= m\ddot{z}, \end{aligned}\right\} \qquad (10\cdot3)$$

where X, Y and Z are the scalar components of $\mathbf{F}$ along the coordinate axes.

11. D'Alembert's principle

In the fundamental equation $F = m\mathrm{a}$, the quantity $m\mathrm{a}$ is called the *effective force* on m. The quantity $m(-\mathbf{a})$ is called the *reversed effective force* or *inertia force*. D'Alembert's principle is based on Newton's second and third laws of motion and states that:

The inertia force is in equilibrium with the external applied force, or

$$\mathbf{F} + m(-\mathbf{a}) = 0.$$

This principle has the effect of reducing a dynamical problem to a problem in statics and may thus make it easier to solve. Centrifugal force is a familiar example of an inertia force.

12. Moment of a force about a point

Let O denote a point in three-dimensional space, let $\mathbf{F}$ denote a force, and let $\mathbf{r}$ be a vector drawn from O to the point of application of $\mathbf{F}$, as indicated in Fig. 13. Then the magnitude of the moment of $\mathbf{F}$ about O is defined to be the product of the force by the perpendicular distance from O to the line of action of $\mathbf{F}$, or

$$\text{magnitude of moment} = (r \sin\theta)\, F.$$

The moment can therefore be represented by the vector product $\mathbf{r} \times \mathbf{F}$, and we thus have the vector equation

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}, \qquad (12\cdot1)$$

where $\mathbf{M}$ denotes the moment of $\mathbf{F}$ about O. The moment vector $\mathbf{M}$ is therefore perpendicular to the plane of $\mathbf{r}$ and $\mathbf{F}$ and pointing in the direction a right-handed screw would advance when rotated in the direction required for rotating $\mathbf{r}$ into the direction of $\mathbf{F}$. In Fig. 13 the vector $\mathbf{M}$ would be directed vertically upward if the plane of $\mathbf{r}$ and $\mathbf{F}$ is horizontal.

Let us take O as the origin of a system of rectangular coordinates.

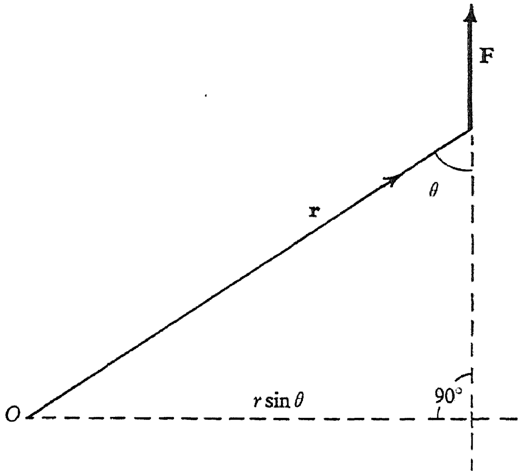


Fig. 13

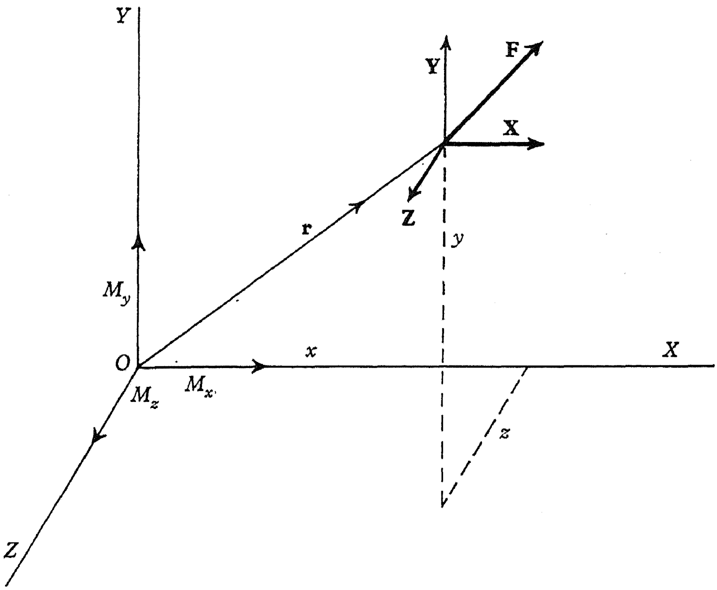


Fig. 14

Then x, y, z are the scalar components of $\mathbf{r}$, X, Y, Z are the scalar components of $\mathbf{F}$, and M_x, M_y, M_z are the scalar components of $\mathbf{M}$. Then the moment of $\mathbf{F}$ about the origin is, by equation (6·8),

$$\mathbf{M}=\mathbf{r}\times\mathbf{F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ X & Y & Z \end{vmatrix}=(yZ-zY)\,\mathbf{i}+(zX-xZ)\,\mathbf{j}+(xY-yX)\,\mathbf{k}. \tag{12·2}$$

Hence
$$\left.\begin{aligned} M_x&=yZ-zY,\\ M_y&=zX-xZ,\\ M_z&=xY-yX.\end{aligned}\right\} \tag{12·3}$$

The quantities M_x, M_y, M_z are the magnitudes of the moments of $\mathbf{F}$ about the x-, y- and z-axes, respectively, as indicated in Fig. 14. It is clear that the component X can have no moment about the x-axis. Viewed from the point O, the moment of Z about the x-axis is positive and the moment of Y is negative, because the moment yZ would tend to produce rotation to the right and the moment zY would tend to produce rotation to the left. Hence the resultant moment about the x-axis is $yZ-zY$. It is to be noted that the moment of $\mathbf{F}$ about each axis is a vector along that axis. It is true in general that *the moment of a force about any axis is a vector along that axis.*

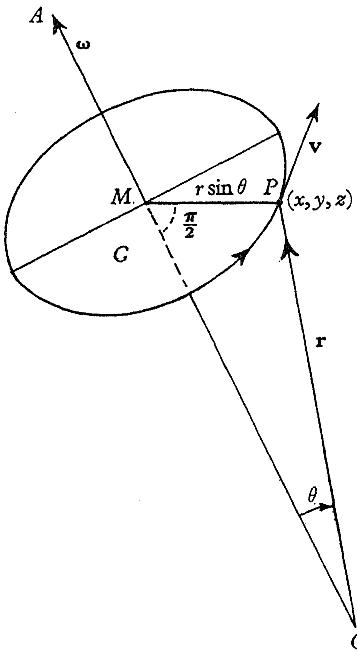


Fig. 15

13. Angular velocity

Angular velocity is a vector quantity and is represented by a vector along the axis of rotation and pointed in the direction required by the right-handed-screw rule.

Consider a rigid body rotating with angular velocity $\boldsymbol{\omega}$ about an instantaneous axis passing through a fixed point O, as indicated in Fig. 15. Let P denote any point in the body, and let $\mathbf{r}$ be a position

vector drawn from O to P. The linear speed of P is evidently $(r\sin\theta)\,\omega$. The linear velocity $\mathbf{v}$ of P is always perpendicular to the plane OMP* and is therefore given by the vector equation

$$\mathbf{v}=\boldsymbol{\omega}\times\mathbf{r}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \omega_1 & \omega_2 & \omega_3\\ x & y & z\end{vmatrix} \tag{13·1}$$

$$=(\omega_2 z-\omega_3 y)\,\mathbf{i}+(\omega_3 x-\omega_1 z)\,\mathbf{j}+(\omega_1 y-\omega_2 x)\,\mathbf{k},$$

where ω_1, ω_2, ω_3 are the scalar components of $\boldsymbol{\omega}$. Hence the scalar components of $\mathbf{v}$ along rectangular axes with origin at O are

$$\left.\begin{aligned} v_x&=\omega_2 z-\omega_3 y,\\ v_y&=\omega_3 x-\omega_1 z,\\ v_z&=\omega_1 y-\omega_2 x.\end{aligned}\right\} \tag{13·2}$$

14. Moment of momentum of a rigid body about a fixed point

Returning to Fig. 15, let us consider a particle of mass m at the point P. The momentum of m is $m\mathbf{v}$, directed along the tangent to the path at P. The *moment* of this momentum about O is, by (12·1),

$$\mathbf{h}=\mathbf{r}\times m\mathbf{v}=m\mathbf{r}\times(\boldsymbol{\omega}\times\mathbf{r}).$$

This vector, it will be observed, is perpendicular to both $\mathbf{r}$ and $\mathbf{v}$. It therefore lies along MP and is directed toward M. The moment of momentum of the entire body about O is therefore

$$\mathbf{H}=\Sigma\mathbf{r}\times m\mathbf{v}=\Sigma m\mathbf{r}\times(\boldsymbol{\omega}\times\mathbf{r})$$

$$=\Sigma m[\boldsymbol{\omega}(\mathbf{r}\cdot\mathbf{r})-\mathbf{r}(\mathbf{r}\cdot\boldsymbol{\omega})]\quad\text{by equation (7·1),}$$

or

$$\mathbf{H}=(\Sigma mr^2)\,\boldsymbol{\omega}-\Sigma m\mathbf{r}(\mathbf{r}\cdot\boldsymbol{\omega}). \tag{14·1}$$

* The proof that $\mathbf{v}$ is perpendicular to the plane OMP and therefore perpendicular to OP is as follows:

For brevity, let C denote the plane of the circle described by P. Then OA, the axis of rotation, is perpendicular to C. The plane OMP is also perpendicular to C; for if a line is perpendicular to a plane, every plane passing through it is perpendicular to that plane.

Now $\mathbf{v}$ is directed along the tangent to the circle at P and therefore lies in the plane C. Moreover, it is perpendicular to the radius MP, which is also the line of intersection of the planes C and OMP. Hence $\mathbf{v}$ is perpendicular to OMP and therefore perpendicular to OP; because if two planes are perpendicular to each other, a straight line drawn in one of them perpendicular to their intersection is perpendicular to the other plane.

This equation shows that $\mathbf{H}$ is the geometric difference of two vectors, one along $\boldsymbol{\omega}$ and the other along $\mathbf{r}$.

If we take a set of rectangular axes with origin at O, then

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

$$\boldsymbol{\omega} = \omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k},$$

$$r^2 = x^2 + y^2 + z^2.$$

Then (14·1) becomes

$$\begin{aligned}
\mathbf{H} &= \Sigma m(x^2+y^2+z^2)(\omega_1\mathbf{i}+\omega_2\mathbf{j}+\omega_3\mathbf{k}) \\
&\quad - \Sigma m(x\omega_1+y\omega_2+z\omega_3)(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}) \\
&= \Sigma m[\omega_1(x^2+y^2+z^2)-(\omega_1 x^2+\omega_2 xy+\omega_3 xz)]\,\mathbf{i} \\
&\quad + \Sigma m[\omega_2(x^2+y^2+z^2)-(\omega_1 xy+\omega_2 y^2+\omega_3 yz)]\,\mathbf{j} \\
&\quad + \Sigma m[\omega_3(x^2+y^2+z^2)-(\omega_1 xz+\omega_2 yz+\omega_3 z^2)]\,\mathbf{k}. \qquad (14\cdot 2)
\end{aligned}$$

Hence the scalar components of $\mathbf{H}$ are

$$\left.\begin{aligned}
H_x &= \Sigma m[\omega_1(x^2+y^2+z^2)-(\omega_1 x^2+\omega_2 xy+\omega_3 xz)] \\
&= \omega_1\Sigma m(y^2+z^2)-\omega_2\Sigma mxy-\omega_3\Sigma mxz, \\
H_y &= \omega_2\Sigma m(x^2+z^2)-\omega_1\Sigma mxy-\omega_3\Sigma myz, \\
H_z &= \omega_3\Sigma m(x^2+y^2)-\omega_1\Sigma mxz-\omega_2\Sigma myz.
\end{aligned}\right\} \qquad (14\cdot 3)$$

The quantities $\Sigma m(y^2+z^2)$, $\Sigma m(x^2+z^2)$ and $\Sigma m(x^2+y^2)$ are the *moments of inertia* of the body about the x-, y- and z-axes, respectively; and the quantities Σmyz, Σmzx and Σmxy are the *products of inertia* about the same axes.

Following the usual custom, we put

$$\left.\begin{aligned}
&A=\Sigma m(y^2+z^2), \quad B=\Sigma m(z^2+x^2), \quad C=\Sigma m(x^2+y^2), \\
&D=\Sigma myz, \quad E=\Sigma mzx, \quad F=\Sigma mxy.
\end{aligned}\right\} \qquad (14\cdot 4)$$

Then (14·3) becomes

$$\left.\begin{aligned}
H_x &= A\omega_1 - F\omega_2 - E\omega_3, \\
H_y &= B\omega_2 - F\omega_1 - D\omega_3, \\
H_z &= C\omega_3 - E\omega_1 - D\omega_2.
\end{aligned}\right\} \qquad (14\cdot 5)$$

If the rigid body possesses three mutually perpendicular planes of symmetry which intersect at the origin, the products of inertia are zero. To see the truth of this statement, assume that the yz-plane is a plane of symmetry. Then for each product such as mxy there is a symmetrically placed product $m(-x)y$ on the other side of the plane, and all these pairs cancel each other in the summation. A similar result holds for each of the other two planes of symmetry.

The lines of intersection of the three planes of symmetry are called the *principal axes* of inertia of the body. Hence, when the coordinate axes are the principal axes of inertia of the body, there are no product-of-inertia terms in equations (14·5) and these equations then become

$$H_x = A\omega_1, \quad H_y = B\omega_2, \quad H_z = C\omega_3, \tag{14·6}$$

and therefore

$$H = \sqrt{(H_x^2 + H_y^2 + H_z^2)} = \sqrt{(A^2\omega_1^2 + B^2\omega_2^2 + C\omega_3^2)}. \tag{14·7}$$

15. The equations of motion of a rigid body about a fixed point

Differentiating the equation $\mathbf{H} = \Sigma(\mathbf{r} \times m\mathbf{v})$ with respect to time, we have

$$\frac{d\mathbf{H}}{dt} = \frac{d}{dt}\Sigma(\mathbf{r} \times m\mathbf{v}) = \Sigma\left(\mathbf{r} \times m\frac{d\mathbf{v}}{dt} + \frac{d\mathbf{r}}{dt} \times m\mathbf{v}\right)$$
$$= \Sigma[(\mathbf{r} \times m\mathbf{a}) + (\mathbf{v} \times m\mathbf{v})].$$

But $m\mathbf{a} = \mathbf{F}$ and $\mathbf{v} \times \mathbf{v} = 0$. Hence the above relation becomes

$$\frac{d\mathbf{H}}{dt} = \Sigma(\mathbf{r} \times \mathbf{F}) = \mathbf{M} \quad \text{by (12·1)}$$
$$= M_x\mathbf{i} + M_y\mathbf{j} + M_z\mathbf{k}, \tag{15·1}$$

where $\mathbf{M}$ denotes the sum of the moments about O of all the external forces acting on the body.

From (15·1) and (12·3), we get the following equations:

$$\left.\begin{aligned} \frac{dH_x}{dt} &= \Sigma(yZ - zY) = M_x, \\ \frac{dH_y}{dt} &= \Sigma(zX - xZ) = M_y, \\ \frac{dH_z}{dt} &= \Sigma(xY - yX) = M_z. \end{aligned}\right\} \tag{15·2}$$

The vector form of these equations is

$$\frac{d\mathbf{H}}{dt}=\mathbf{M},$$

and it refers to *fixed space.*

Equations (15·1) and (15·2) are the mathematical statements of a most important dynamical law:

The time rate of change of the angular momentum of a rigid body rotating about any axis is equal to the moment of the applied external forces about the same axis.*

Since $d\mathbf{M}/dt$ is the time rate of change of $\mathbf{H}$, it may be regarded as the velocity of the end of the vector $\mathbf{H}$. Equations (15·1) and (15·2) therefore state:

The velocity of the end of the angular-momentum vector is equal to the moment of the external forces.

This statement of the law is sometimes convenient to use in dealing with the gyroscope.

If no external forces are acting on the body, or if there are such forces but they have no moment about the axis of rotation, then $\mathbf{M}=0$ and (15·1) becomes

$$\frac{d\mathbf{H}}{dt}=0.$$

Hence $$\mathbf{H}=\text{a constant}. \qquad (15{\cdot}3)$$

Note that in this case the angular-momentum vector remains constant in magnitude and fixed in direction.

Equation (15·3) expresses the law of the Conservation of Angular Momentum.

Furthermore, when $\mathbf{H}$ is constant equation (14·7) gives

$$A^2\omega_1^2+B^2\omega_2^2+C^2\omega_3^2=\text{a constant}. \qquad (15{\cdot}4)$$

16. The general differential equations of motion. Euler's equations

In the previous article the equations of motion were referred to a set of axes fixed in space. In order to study the general motion of a rigid body about a fixed point, however, it is necessary to employ two sets of rectangular axes, a fixed set and a moving set, but with both sets having the fixed point as origin. Let O (Fig. 16) denote the

* Moment of momentum is also called angular momentum and will often be called that in this book, because the name 'angular' is less clumsy than 'moment of'.

fixed point, let $O-X_1Y_1Z_1$ denote a set of fixed axes (fixed in space), and let $O-XYZ$ denote another set of rectangular axes which moves as a rigid trihedral framework about O in any manner.

Let $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ denote unit vectors along OX, OY and OZ, respectively; let $\boldsymbol{\omega}$ denote the angular velocity of the moving trihedron

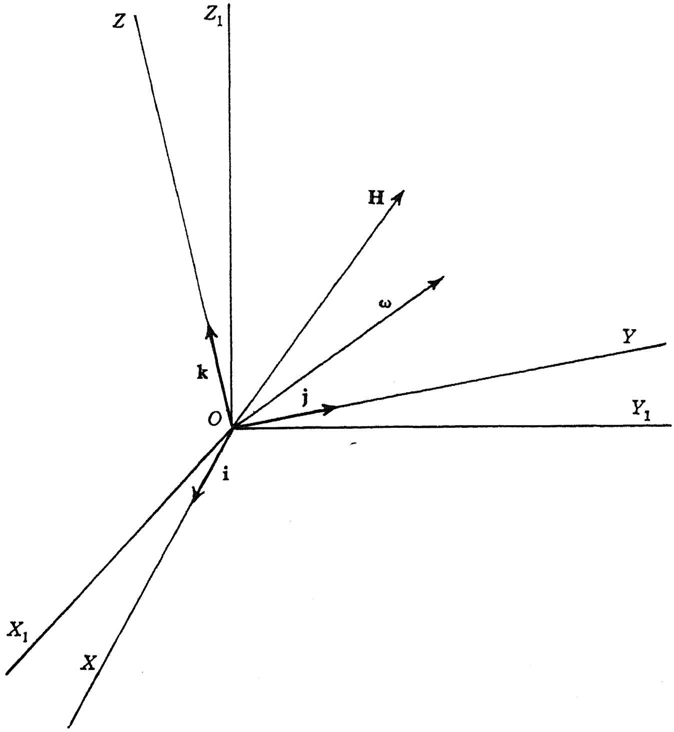


Fig. 16

with respect to (as viewed from) the fixed set of axes; and let $\mathbf{H}$ denote the angular momentum of the body with respect to the fixed axes. Then if ω_x, ω_y, ω_z and H_x, H_y, H_z denote the magnitudes of the projections of $\boldsymbol{\omega}$ and $\mathbf{H}$ on the axes of the moving trihedron, we have

$$\boldsymbol{\omega}=\omega_x\mathbf{i}+\omega_y\mathbf{j}+\omega_z\mathbf{k}, \tag{16·1}$$

$$\mathbf{H}=H_x\mathbf{i}+H_y\mathbf{j}+H_z\mathbf{k}. \tag{16·2}$$

Although $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors along OX, OY, OZ and are constants with respect to those axes, they are not constants with

respect to the fixed axes $O-X_1Y_1Z_1$. Their directions change with respect to (as viewed from) the fixed axes and hence vary with the time. The time derivative of $\mathbf{H}$ is therefore from (16·2):

$$\frac{d\mathbf{H}}{dt}=\mathbf{i}\frac{dH_x}{dt}+\mathbf{j}\frac{dH_y}{dt}+\mathbf{k}\frac{dH_z}{dt}+H_x\frac{d\mathbf{i}}{dt}+H_y\frac{d\mathbf{j}}{dt}+H_z\frac{d\mathbf{k}}{dt}. \qquad (16\cdot3)$$

To evaluate the derivatives $d\mathbf{i}/dt$, $d\mathbf{j}/dt$, $d\mathbf{k}/dt$ we regard $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ as position vectors drawn from O. If $\mathbf{r}$ is a position vector drawn from O to any point in a rotating body, then by (13·1)

$$\frac{d\mathbf{r}}{dt}=\boldsymbol{\omega}\times\mathbf{r}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \omega_x & \omega_y & \omega_z\\ r_x & r_y & r_z\end{vmatrix}.$$

Now applying this formula to the vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ in turn, we have

$$\frac{d\mathbf{i}}{dt}=\boldsymbol{\omega}\times\mathbf{i}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \omega_x & \omega_y & \omega_z\\ 1 & 0 & 0\end{vmatrix}=\omega_z\mathbf{j}-\omega_y\mathbf{k},$$

$$\frac{d\mathbf{j}}{dt}=\boldsymbol{\omega}\times\mathbf{j}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \omega_x & \omega_y & \omega_z\\ 0 & 1 & 0\end{vmatrix}=-\omega_z\mathbf{i}+\omega_x\mathbf{k},$$

$$\frac{d\mathbf{k}}{dt}=\boldsymbol{\omega}\times\mathbf{k}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \omega_x & \omega_y & \omega_z\\ 0 & 0 & 1\end{vmatrix}=\omega_y\mathbf{i}-\omega_x\mathbf{j}.$$

On substituting these derivatives in (16·3) we get

$$\frac{d\mathbf{H}}{dt}=\left(\frac{dH_x}{dt}-H_y\omega_z+H_z\omega_y\right)\mathbf{i}$$
$$+\left(\frac{dH_y}{dt}+H_x\omega_z-H_z\omega_x\right)\mathbf{j}$$
$$+\left(\frac{dH_z}{dt}-H_x\omega_y+H_y\omega_x\right)\mathbf{k}.$$

But by (15·1) $$\frac{d\mathbf{H}}{dt}=\mathbf{M}=M_x\mathbf{i}+M_y\mathbf{j}+M_z\mathbf{k}.$$

On equating these two values of $d\mathbf{H}/dt$, we get

$$\left.\begin{aligned}\frac{dH_x}{dt}+H_z\omega_y-H_y\omega_z&=M_x,\\ \frac{dH_y}{dt}+H_x\omega_z-H_z\omega_x&=M_y,\\ \frac{dH_z}{dt}+H_y\omega_x-H_x\omega_y&=M_z.\end{aligned}\right\}\tag{16·4}$$

These are the differential equations of motion referred to the moving axes, that is, *referred to the axes of the moving trihedron*. They are the most important differential equations connected with gyroscopic theory.

If the axes OX, OY, OZ coincide with the principal axes of inertia of the body for the point O, then

$$H_x=A\omega_x,\quad H_y=B\omega_y,\quad H_z=C\omega_z,$$

and therefore

$$\frac{dH_x}{dt}=A\frac{d\omega_x}{dt},\quad \frac{dH_y}{dt}=B\frac{d\omega_y}{dt},\quad \frac{dH_z}{dt}=C\frac{d\omega_z}{dt}.$$

Substituting these into (16·4), we get

$$\left.\begin{aligned}A\frac{d\omega_x}{dt}+(C-B)\,\omega_y\omega_z&=M_x,\\ B\frac{d\omega_y}{dt}+(A-C)\,\omega_z\omega_x&=M_y,\\ C\frac{d\omega_z}{dt}+(B-A)\,\omega_x\omega_y&=M_z.\end{aligned}\right\}\tag{16·5}$$

If the rotating body is dynamically symmetrical about the axis OZ, then $B=A$ and equations (16·5) become

$$\left.\begin{aligned}A\frac{d\omega_x}{dt}+(C-A)\,\omega_y\omega_z&=M_x,\\ A\frac{d\omega_y}{dt}+(A-C)\,\omega_z\omega_x&=M_y,\\ C\frac{d\omega_z}{dt}\qquad\qquad\qquad&=M_z.\end{aligned}\right\}\tag{16·6}$$

Equations (16·5) and their special cases (16·6) are known as *Euler's dynamical equations* for the motion of a rigid body about a fixed point.

Here we make a short digression to point out an interesting phenomenon. Assume that we have a solid of revolution which can be made to spin in a vertical position about its geometric axis. Then its differential equation of motion would be the third equation of (16·6). Suppose the solid is given a vigorous spin and let go. Then its equation of motion after that instant would be

$$C\frac{d\omega_z}{dt}=0,$$

from which $$C\omega_z = \text{a constant} = k, \quad \text{say},$$

or $$\omega_z = \frac{k}{C}.$$

This equation shows that if the solid is of such construction that its moment of inertia can be increased or decreased at will, its angular velocity will immediately slow down or speed up so as to keep the product $C\omega$ constant. This is an example of the principle of conservation of angular momentum; for, by (14·6), $C\omega$ is the angular momentum in this case. Acrobats and other actors frequently utilize this principle to increase or decrease the spin of their bodies, which they can do by drawing in or spreading their arms or legs.

The vector form of equations (16·4) is

$$\frac{d\mathbf{H}}{dt} + \boldsymbol{\omega} \times \mathbf{H} = \mathbf{M}, \tag{16·7}$$

as may be verified by replacing $\mathbf{H}$ and $\boldsymbol{\omega}$ by their equivalent forms (16·2) and (16·1), respectively. In (16·7) $\mathbf{M}$ is the moment about O of the external forces acting on the body.

It is to be noted that (16·7) is the differential equation of motion *referred to the axes of the moving trihedron.*

If $\mathbf{H}$ is constant, then $d\mathbf{H}/dt = 0$ and (16·7) becomes

$$\boldsymbol{\omega} \times \mathbf{H} = \mathbf{M}. \tag{16·8}$$

The vector product $\boldsymbol{\omega} \times \mathbf{H}$ is sometimes called the gyroscopic couple. It is also called the centrifugal couple. This vector is perpendicular to both $\boldsymbol{\omega}$ and $\mathbf{H}$ and is therefore perpendicular to the plane of those vectors. Hence in this case $\mathbf{M}$ is perpendicular to the plane of $\boldsymbol{\omega}$ and $\mathbf{H}$.

If the moment of the external applied forces is zero, then (16·7) becomes

$$\frac{d\mathbf{H}}{dt}=\mathbf{H}\times\boldsymbol{\omega}. \tag{16·9}$$

Finally, if $\boldsymbol{\omega}\times\mathbf{H}=0$, which would be the case if $\boldsymbol{\omega}$ and $\mathbf{H}$ should coincide, then (16·7) becomes

$$\frac{d\mathbf{H}}{dt}=\mathbf{M}. \tag{16·10}$$

Attention is called to the dual utility of equations (16·4), (16·5) and (16·6) in the solution of gyroscopic problems:

(1) If the external forces acting on the gyroscope are known, the moments M_x, M_y, M_z can be found. Then the right-hand members of the equations are known, and the motion of the gyroscope can be determined by integrating the resulting differential equations.

(2) If the motion of the gyroscope is given, the left-hand members of the equations will be known and then the moments M_x, M_y, M_z which could produce the given motion can be determined by substituting in the left-hand members the assumed or given data concerning the motion.

These two uses will be shown in the next chapter.

17. Kinetic energy of the body

If a rigid body is rotating with angular velocity $\boldsymbol{\omega}$ about an instantaneous axis passing through a fixed point O, and $\mathbf{r}$ is a position vector drawn from O to a particle of mass m whose coordinates are (x, y, z), the linear velocity of the particle is given by the formula $\mathbf{v}=\boldsymbol{\omega}\times\mathbf{r}=v_x\mathbf{i}+v_y\mathbf{j}+v_z\mathbf{k}$ by (13·1),

and the scalar components v_x, v_y, v_z are given by (13·2). The kinetic energy of the body is given by the formula

$$T=\tfrac{1}{2}\Sigma mv^2=\tfrac{1}{2}\Sigma m(v_x^2+v_y^2+v_z^2).$$

Replacing v_x^2, v_y^2, v_z^2 by their values as given in (13·2) we obtain

$$T=\tfrac{1}{2}\Sigma m[(\omega_2 z-\omega_3 y)^2+(\omega_3 x-\omega_1 z)^2+(\omega_1 y-\omega_2 x)^2],$$

or

$$2T=\omega_1^2\Sigma m(y^2+z^2)+\omega_2^2\Sigma m(z^2+x^2)+\omega_3^2\Sigma(x^2+y^2)$$
$$-2\omega_1\omega_2\Sigma mxy-2\omega_1\omega_3\Sigma mzx-2\omega_2\omega_3\Sigma myz.$$

Now if the principal axes of inertia are taken as coordinate axes, the products of inertia disappear and we then have, by (14·4),

$$T=\tfrac{1}{2}(A\omega_1^2+B\omega_2^2+C\omega_3^2). \tag{17·1}$$

From (16·1), (16·2) and (14·6) we have

$$\boldsymbol{\omega}\cdot\mathbf{H}=A\omega_x^2+B\omega_y^2+C\omega_z^2=2T \quad \text{by (17·1).} \qquad (17·2)$$

Hence the kinetic energy is greatest when $\mathbf{H}$ is parallel to $\boldsymbol{\omega}$. This is the same case as $\boldsymbol{\omega}\times\mathbf{H}=0$.

Since $\boldsymbol{\omega}\cdot\mathbf{H}=\omega H\cos\theta$, we have, from (17·2),

$$\omega H\cos\theta=2T. \qquad (17·3)$$

Then since ω, H and T are all positive quantities, *θ must be an acute angle*, either positive or negative.

From (17·1) and (14·6) we obtain the further relations:

$$\left.\begin{aligned}\frac{\partial T}{\partial\omega_x}&=A\omega_x=H_x,\\ \frac{\partial T}{\partial\omega_y}&=B\omega_y=H_y,\\ \frac{\partial T}{\partial\omega_z}&=C\omega_z=H_z.\end{aligned}\right\} \qquad (17·4)$$

If the moments of the external forces are zero, then Euler's equations (16·5) become

$$\left.\begin{aligned}A\frac{d\omega_1}{dt}+(C-B)\,\omega_2\omega_3&=0,\\ B\frac{d\omega_2}{dt}+(A-C)\,\omega_1\omega_3&=0,\\ C\frac{d\omega_3}{dt}+(B-A)\,\omega_1\omega_2&=0.\end{aligned}\right\} \qquad (17·5)$$

On multiplying these equations by ω_1, ω_2 and ω_3, respectively, and adding the results, we get

$$A\omega_1\frac{d\omega_1}{dt}+B\omega_2\frac{d\omega_2}{dt}+C\omega_3\frac{d\omega_3}{dt}=0.$$

Integrating this, we have

$$A\omega_1^2+B\omega_2^2+C\omega_3^2=\text{a constant}=2T \quad \text{by (17·1).}$$

Hence the kinetic energy is constant when the moments of the external forces are zero. A rigid body rotating about an instantaneous axis through its center of gravity and acted on by no other forces than its own weight, would come under this case.

18. Euler's angles

The orientation of a rigid body moving about a fixed point is best fixed by three angles θ, ϕ and ψ, called Euler's angles. These connect the moving axes to the fixed axes and are shown in Fig. 17.

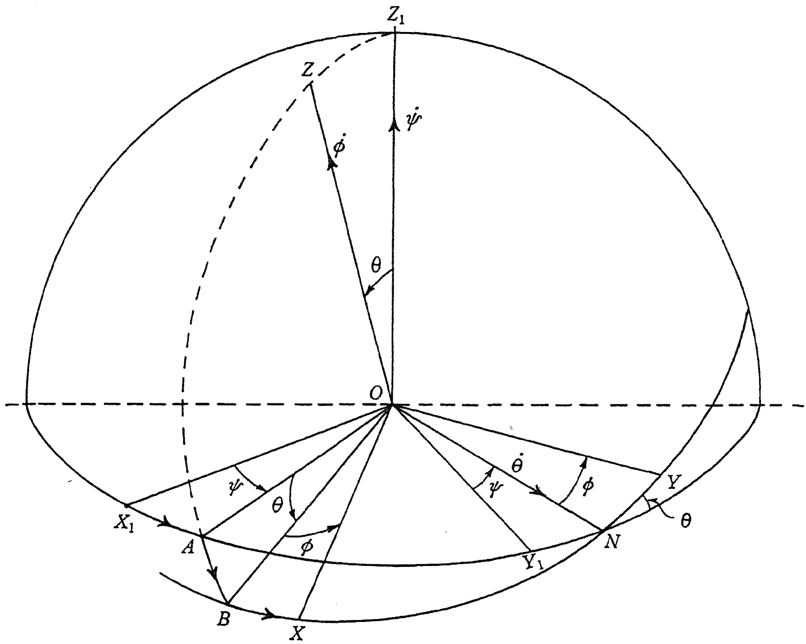


Fig. 17

The moving trihedron $O-XYZ$ can be brought from the fixed position $O-X_1Y_1Z_1$ to the position shown in Fig. 17 by the following procedure, carried out in the order stated:

(1) Rotate through angle ψ about OZ_1, thereby bringing OY_1 into coincidence with ON.

(2) Rotate through angle θ about ON, thus bringing OZ to the position shown in Fig. 17.

(3) Rotate through angle ϕ about OZ, thus bringing OX to the position shown in the figure.

19. Components of the angular velocity in terms of Euler's angles

To find the values of ω_1, ω_2, ω_3 in terms of θ, ϕ, ψ and their time derivatives, we consider the projections of all the angular-velocity vectors on each of the moving axes. In Fig. 18 the plane

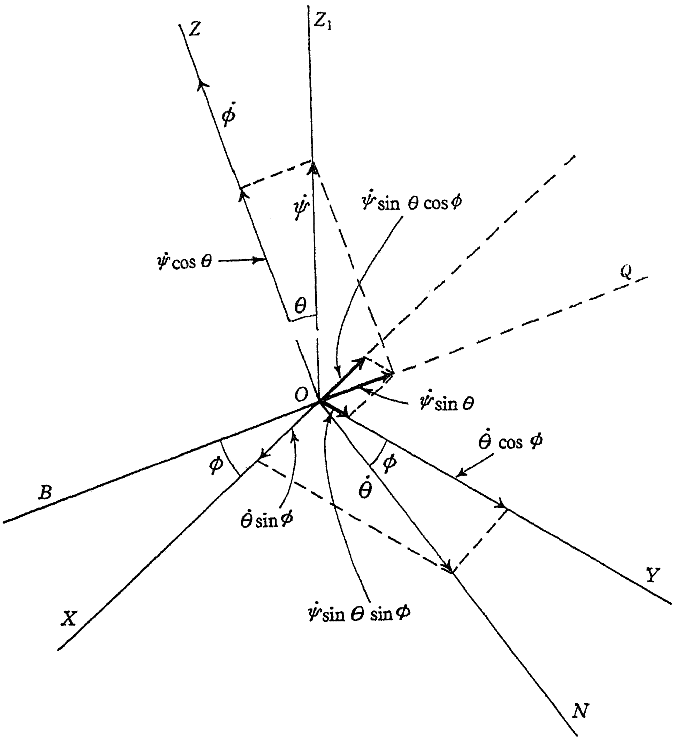


Fig. 18

OZ_1Z is perpendicular to ON, and BQ is the intersection of the plane OZ_1Z and the XY-plane. This line BQ is to be used as a bridge for transferring vectors from the OZ_1Z-plane to the XY-plane.

Inspection of Fig. 18 shows that the projection of $\dot\psi$ on BQ is $\dot\psi\sin\theta$, and this projection therefore has components $-\dot\psi\sin\theta\cos\phi$ along OX and $\dot\psi\sin\theta\sin\phi$ along OY. The projections of $\dot\theta$ on OX and OY are $\dot\theta\sin\phi$ and $\dot\theta\cos\phi$, respectively. If $\boldsymbol{\omega}$ denotes the

angular velocity of the moving trihedron $O-XYZ$ about an instantaneous axis through O, the components $\omega_x, \omega_y, \omega_z$ along the moving axes are therefore

$$\left.\begin{aligned} \boldsymbol{\omega}_x &= \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi, \\ \boldsymbol{\omega}_y &= \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi, \\ \boldsymbol{\omega}_z &= \dot{\phi} + \dot{\psi}\cos\theta. \end{aligned}\right\} \tag{19·1}$$

These equations are sometimes called Euler's kinematical equations.

The kinetic energy of the body in terms of θ, ϕ, ψ and their time derivatives is, by (17·1),

$$\begin{aligned} T = \tfrac{1}{2}[A(\dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi)^2 \\ + B(\dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi)^2 + C(\dot{\phi} + \dot{\psi}\cos\theta)^2]. \end{aligned} \tag{19·2}$$

For a solid of revolution about the z-axis, we have $B = A$ and then (19·2) reduces to

$$T = [A(\dot{\theta}^2 + \dot{\psi}^2\sin^2\theta) + C(\dot{\phi} + \dot{\psi}\cos\theta)^2]. \tag{19·3}$$

20. Rotation of a rigid body about a fixed axis

The fundamental vector equation (15·1) and its Cartesian equivalents (15·2) hold for the motion of a rigid body about any axis passing through a fixed point. They therefore hold equally well for a *fixed* axis passing through the fixed point.

Let us take the x-axis as the fixed axis. Then from (14·3) we have

$$H_x = \omega_x \Sigma m(y^2 + z^2) - \omega_y \Sigma mxy - \omega_z \Sigma mxz.$$

But since the rotation is entirely about the x-axis, ω_y and ω_z are zero. Hence in this case

$$H_x = \omega_x \Sigma m(y^2 + z^2) = \omega_x I_x,$$

or, for any line, $\qquad H = I\omega.$

Then, by (15·1) or (15·2),

$$\frac{dH}{dt} = I\frac{d\omega}{dt} = M. \tag{20·1}$$

If θ is the angle through which the body has rotated from a given position, then $\omega = d\theta/dt$ and (20·1) becomes

$$I\frac{d^2\theta}{dt^2} = M, \tag{20·2}$$

or

$$I\ddot{\theta} = M. \tag{20·3}$$

This equation, in any of the three forms just given, is the fundamental equation for the rotation of a rigid body about a fixed axis. It corresponds to the fundamental equation of Art. 10 for rectilinear motion.

CHAPTER III

Theory of the Gyroscope

21. Definition and behavior of a gyroscope

A gyroscope may be broadly defined as a solid body capable of rotating at high angular velocity about an instantaneous axis which always passes through a fixed point. The fixed point may be the center of gravity of the solid or it may be any other point.

More specifically, however, the usual form of gyroscope is a mechanical device the essential part of which is a flywheel having a heavy rim and so mounted that, while spinning at high speed, its axis of rotation can turn in any direction about a fixed point on that axis. The flywheel can be mounted so as to turn about its center of gravity by means of double gimbals called a Cardan suspension (Fig. 19).

The familiar toy top and 'spool dancer' are forms of gyroscopes in which the fixed point is the point of contact of the top with the floor or table on which it spins.

A rapidly spinning flywheel, mounted as described above, resists any effort to change the direction of its axis of spin. In other words, the flywheel persists in maintaining its plane of rotation. For example, if the axis of spin is horizontal and we apply a steady downward force to the inner ring at one end of the spin axle, the end of the axle is not carried downward as we should expect but begins to skew around in a horizontal plane. This skewing around of the spin axis is called *precession.* It arises in obedience to the fundamental relation $d\mathbf{H}/dt = \mathbf{M}$. Since the applied external moment cannot change the rate of spin of the flywheel, it increases the angular momentum of the mounted wheel and ring by making it rotate about a vertical axis.

The direction of this precession about a vertical axis depends on the direction of spin of the flywheel and also on which end of the axle is pressed downward. When the wheel is spinning in a given

direction, a downward pressure on one end of the axle will cause the axle to precess in one direction, whereas a downward pressure on the other end of the axle will cause precession in the opposite direction.

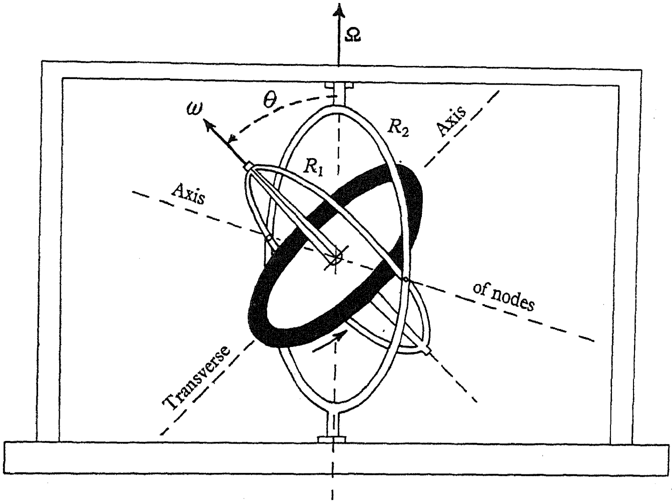


Fig. 19

22. Steady or regular precession or gyro

Let us now assume that a heavy flywheel* is mounted so that its axis of spin always passes through the center of gravity of the gyro (~~flywheel~~), as indicated in Fig. 20. Let us further assume that the gyro is spinning with *constant* angular velocity $\dot{\phi}$ about the axis OZ, that OZ makes a *constant* angle θ with the vertical axis OZ_1, and that the axis of spin precesses about OZ_1 with *constant* angular velocity Ω. Constant precession under the conditions just described is called *steady precession.* It is required to find what external moment is necessary to *maintain* such steady precession after once started.

As the spin axis precesses around OZ_1 with constant angular velocity Ω, the plane ZOZ_1 (~~called the *azimuthal plane* or *precessional plane*~~) also rotates about OZ_1 with angular velocity Ω. Hence if we take the rectangular trihedron $O-XYZ$ as the system

* The mounted flywheel in a gyroscope is frequently called a *gyro* and will often be called that in this book.

of moving axes, the angular velocity of the trihedron about O is also Ω.* Hence the components of this angular velocity along the moving axes are

$$\Omega_x = -\Omega \sin\theta, \quad \Omega_y = 0, \quad \Omega_z = \Omega\cos\theta$$

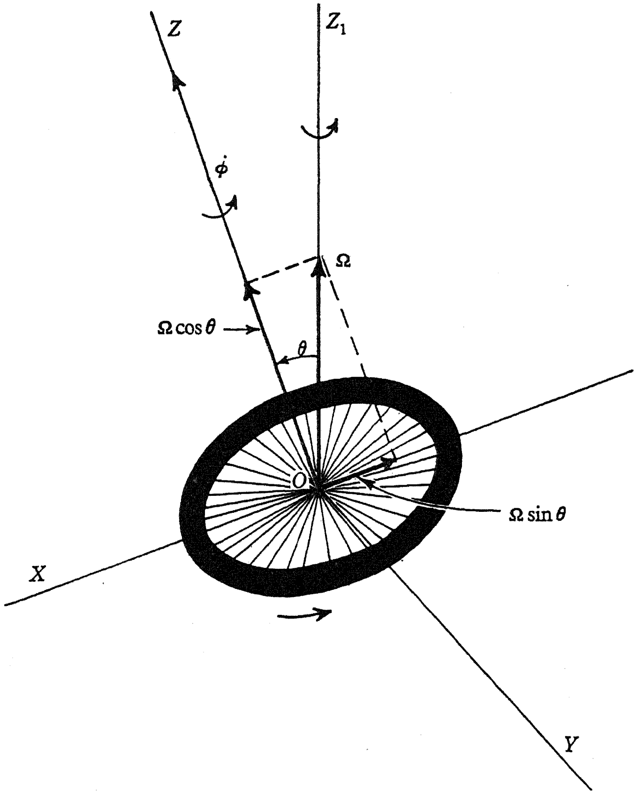


Fig. 20

If C denotes the moment of inertia of the gyro about its axis of spin and A denotes the moment of inertia about the transverse axis OX, then, since OZ and OX are principal axes of inertia for the center of gravity O and since the total angular velocity vector along OZ is $\Omega\cos\theta + \dot{\phi}$, we have

$$H_x = -A\Omega\sin\theta, \quad H_y = 0, \quad H_z = C(\dot{\phi} + \Omega\cos\theta).$$

* Note that the axes of the trihedron are not fixed in the gyro.

Now since θ, $\dot{\phi}$ and Ω are all constants, we get

$$\frac{dH_x}{dt}=0, \quad \frac{dH_y}{dt}=0, \quad \frac{dH_z}{dt}=0.$$

Substituting the above values into (16·4), we get

$$\left.\begin{aligned} M_x &= 0, \\ M_y &= [C\dot{\phi}+(C-A)\,\Omega\cos\theta]\,\Omega\sin\theta, \\ M_z &= 0. \end{aligned}\right\} \qquad (22\cdot 1)$$

The required moment is thus entirely about the y-axis, or in the azimuthal plane ZOZ_1.

The expression for M_y can be rearranged and written in the following equivalent form:

$$\begin{aligned} M_y &= \left[C+(C-A)\frac{\Omega}{\dot{\phi}}\cos\theta\right]\Omega\dot{\phi}\sin\theta \\ &= \left[C+(C-A)\frac{\Omega}{\dot{\phi}}\cos\theta\right](\mathbf{\Omega}\times\dot{\boldsymbol{\phi}}). \end{aligned} \qquad (22\cdot 2)$$

As $C>A$ for a flywheel of the usual type,* the quantity in brackets in (22·2) is positive. Hence M_y is a positive quantity and as a vector it would be drawn from O along the positive y-axis. Then, by the right-handed-screw rule, this moment tends to increase the angle θ. Hence for steady precession with a constant θ there must be a turning moment in the azimuthal plane, and this moment must tend to increase θ.

When a gyroscope is spinning at high speed, the precessional velocity Ω is negligible in comparison with the spin velocity $\dot{\phi}$. Hence in that case we may neglect the fraction $\Omega/\dot{\phi}$. Then (22·2) becomes

$$M_y = C\Omega\dot{\phi}\sin\theta = C\,(\mathbf{\Omega}\times\dot{\boldsymbol{\phi}}). \qquad (22\cdot 3)$$

* For a flat disk, $C=\frac{1}{2}mr^2$ and $A=\frac{1}{4}mr^2$; and for a wheel in which most of the mass is in the rim, $C=mr^2$, approximately, and $A=\frac{1}{2}mr^2$, approximately. Hence, for the usual rotor of a gyroscope, $C>A$.

If $\theta = 90°$, this becomes

$$M_y = C\Omega\dot{\phi} = C(\boldsymbol{\Omega} \times \dot{\boldsymbol{\phi}}). \tag{22·4}$$

This special equation (22·4) is of great practical importance. When written in the form

$$\Omega = \frac{M_y}{C\dot{\phi}}, \tag{22·4a}$$

it states that the velocity of precession varies directly with the external moment and inversely as the velocity of spin of the gyro. Thus, for a given external moment the greater the spin velocity, the slower the precession.

Another aspect of (22·4) will be discussed in Art. 24.

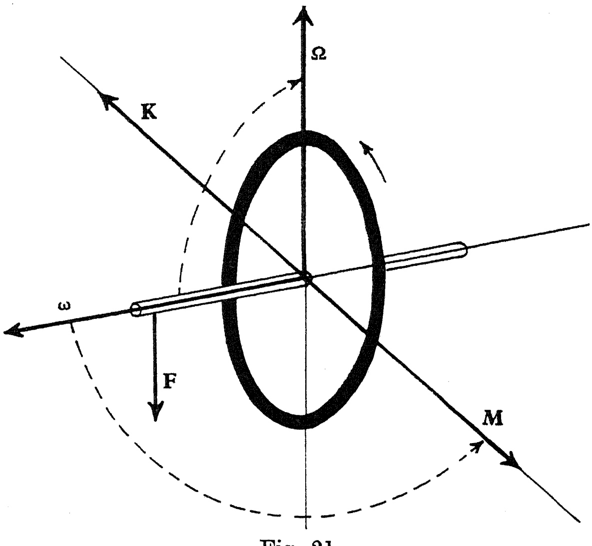


Fig. 21

23. Direction of precession

In practical applications of the gyroscope, it is of the utmost importance to know the direction in which the spin axis will precess under given conditions. In the highly important case where $\theta = 90°$ and $M = C\Omega\dot{\phi}$, reference to Fig. 21 will show that the spin axis OZ will lie along OX. Then, according to the right-handed-screw rule, a precession to the right about OZ_1 will swing OX around toward OY; that is, it will swing the spin vector around toward the moment vector, as indicated more clearly in Fig. 21.

Hence we can state the following rule for determining the direction of precession:

An applied external moment causes the positive end of the spin axis to precess toward the positive end of the moment axis, the positive end of each axis being that determined by the right-handed-screw rule.

An extension of this rule is given at the end of Art. 33.

24. Free precession

The formulas of Art. 22 give the external moment required to maintain a steady precession after it is once started. It is conceivable that for some velocity of spin the precession might continue after the external moment is removed. To find such a velocity we put $M_y = 0$ in (22·1). We then get

$$C\dot{\phi} + (C - A)\,\Omega\cos\theta = 0,$$

or
$$\Omega = -\frac{C\dot{\phi}}{(C-A)\cos\theta}. \qquad (24{\cdot}1)$$

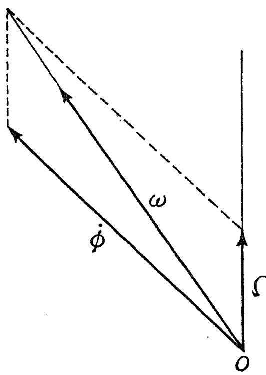


Fig. 22

Equation (24·1) expresses the condition for *free precession*. When this condition is satisfied, a steady precession would continue indefinitely without the action of an external moment if no frictional forces interfered.

The negative sign before the right-hand member of (24·1) should be noted. This sign shows that when $C > A$ and $\theta < 90°$, Ω and $\dot{\phi}$ have opposite signs. This means that the precession would be backward or *retrograde*; whereas if $C < A$, Ω and $\dot{\phi}$ have the same sign and precession would be *direct*.

Equation (24·1) shows another interesting result. If $|C - A| < C$, the fraction $C/(C-A)\cos\theta$ is greater than 1. Hence in that case the precession Ω is greater than the spin $\dot{\phi}$. Furthermore, if θ is near 90°, Ω is very large.

At this point we wish to call attention to the fact that when a gyroscope is precessing, the axis of instantaneous rotation does not coincide with the axis of spin. The total velocity about the instantaneous axis is the geometric or vector sum of the spin velocity and the precessional velocity, as indicated in Fig. 22.

25. Unsteady precession. General motion of a gyroscope

In Art. 22 we discussed the motion of a gyroscope under the conditions that the spin velocity, the precessional velocity and the angle θ were all constants. We now drop these restrictions and

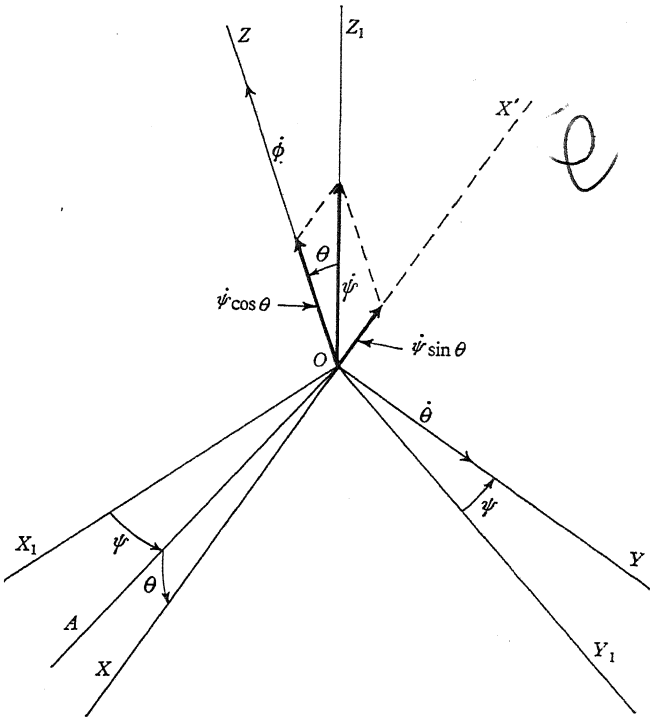


Fig. 23

assume that the angle θ, the spin velocity $\dot{\phi}$ and the precessional velocity Ω may all vary in any manner. We are to find the necessary external moments in this case.

Referring to Fig. 23, let the axis of spin lie along OZ, and let this axis precess about OZ_1 with variable angular velocity $\dot{\psi}$. The time rate of change of θ will be denoted by $\dot{\theta}$ and is seen to be a vector along the positive direction OY. The spin velocity with respect to the azimuthal plane ZOZ_1 is denoted by $\dot{\phi}$ and is laid off as a vector along OZ.

The position of the moving trihedron $O-XYZ$ is given at any

instant by the angles ψ and θ (Fig. 23). The projections of the angular velocity of the trihedron on the moving axes are $-\dot{\psi}\sin\theta$ on the x-axis, $\dot{\theta}$ on the y-axis, and $\dot{\psi}\cos\theta$ on the z-axis, as indicated in the figure. Hence we have

$$\omega_x = -\dot{\psi}\sin\theta, \quad \omega_y = \dot{\theta}, \quad \omega_z = \dot{\psi}\cos\theta. \tag{25·1}$$

Because of the spin velocity along OZ, the total angular velocity along that axis is $\dot{\phi} + \dot{\psi}\cos\theta$.

Now if the axes OX, OY, OZ are principal axes of inertia of the gyro for its center of gravity O, the components of the angular momentum of the gyro along the axes are

$$H_x = -A\dot{\psi}\sin\theta, \quad H_y = A\dot{\theta}, \quad H_z = C(\dot{\phi} + \dot{\psi}\cos\theta).$$

Consequently we have

$$\frac{dH_x}{dt} = -A(\ddot{\psi}\sin\theta + \dot{\theta}\dot{\psi}\cos\theta),$$

$$\frac{dH_y}{dt} = A\ddot{\theta},$$

$$\frac{dH_z}{dt} = C\frac{d}{dt}(\dot{\phi} + \dot{\psi}\cos\theta).$$

On substituting into (16·4) the components of the angular velocity and angular momentum, and the values of the derivatives given above, we get

$$-A(\ddot{\psi}\sin\theta + \dot{\theta}\dot{\psi}\cos\theta) + C(\dot{\phi} + \dot{\psi}\cos\theta)\dot{\theta} - A\dot{\theta}\dot{\psi}\cos\theta = M_x,$$

$$A\ddot{\theta} - A\dot{\psi}^2\sin\theta\cos\theta + C(\dot{\phi} + \dot{\psi}\cos\theta)\dot{\psi}\sin\theta = M_y,$$

$$C\frac{d}{dt}(\dot{\phi} + \dot{\psi}\cos\theta) = M_z.$$

These equations can be simplified somewhat and written in the forms

$$\left.\begin{aligned} -A\ddot{\psi}\sin\theta - 2A\dot{\theta}\dot{\psi}\cos\theta + C(\dot{\phi} + \dot{\psi}\cos\theta)\dot{\theta} &= M_x, \\ A\ddot{\theta} + [C\dot{\phi} + (C-A)\dot{\psi}\cos\theta]\dot{\psi}\sin\theta &= M_y, \\ C\frac{d}{dt}(\dot{\phi} + \dot{\psi}\cos\theta) &= M_z. \end{aligned}\right\} \tag{25·2}$$

These are the fundamental differential equations of motion of a symmetrical gyroscope. It will be seen that they reduce to (22·1) when θ, $\dot{\phi}$ and $\dot{\psi}$ are constants.

It will further be seen on comparing the equation for M_y with equation (22·1) that M_y in the general case equals the moment required for maintaining steady precession plus the accelerating moment $A\ddot{\theta}$.

When a gyro is spinning at high speed, the quantity $\dot{\phi}$ is the dominant term in the left-hand members of equations (25·2), and all other terms are negligible in comparison with it. Hence if we neglect all terms not containing $\dot{\phi}$, those equations become

$$\left.\begin{aligned} C\dot{\theta}\dot{\phi} &= M_x, \\ C\dot{\phi}\dot{\psi}\sin\theta &= M_y, \\ C\ddot{\phi} &= M_z. \end{aligned}\right\} \qquad (25\cdot3)$$

And if the spin velocity is constant, or nearly so, we may neglect $\ddot{\phi}$. Then equations (25·3) become

$$\left.\begin{aligned} C\dot{\theta}\dot{\phi} &= M_x, \\ C\dot{\phi}\dot{\psi}\sin\theta &= M_y. \end{aligned}\right\} \qquad (25\cdot4)$$

In some applications of the gyroscope there is no external moment about OZ; in those cases $M_z = 0$. Then when the axes of the moving trihedron coincide with the principal axes of the gyro, we have

$$H_x = A\omega_x, \quad H_y = A\omega_y, \quad H_z = C(\dot{\phi} + \omega_z)$$

and

$$\frac{dH_x}{dt} = A\dot{\omega}_x, \quad \frac{dH_y}{dt} = A\dot{\omega}_y, \quad \frac{dH_z}{dt} = C\frac{d}{dt}(\dot{\phi} + \omega_z).$$

On substituting these quantities into (16·4), we get

$$\begin{aligned} A\dot{\omega}_x + C\omega_y(\dot{\phi} + \omega_z) - A\omega_y\omega_z &= M_x, \\ A\dot{\omega}_y + A\omega_x\omega_z - C\omega_x(\dot{\phi} + \omega_z) &= M_y, \\ C\frac{d}{dt}(\dot{\phi} + \omega_z) &= 0. \end{aligned}$$

From the third of these equations we get

$$\dot{\phi} + \omega_z = \text{a constant} = \Omega, \quad \text{say.}$$

Now substituting this value of $\dot{\phi} + \omega_z$ into the first two equations, we have

$$\left.\begin{aligned} A\dot{\omega}_x + C\Omega\omega_y - A\omega_y\omega_z &= M_x, \\ A\dot{\omega}_y + A\omega_x\omega_z - C\Omega\omega_x &= M_y. \end{aligned}\right\} \qquad (25\cdot5)$$

Here it is to be noted that Ω represents the *total* angular velocity about OZ, the spin velocity plus the z-component of the angular velocity of the moving trihedron.

26. Steady precession of an unsymmetrical gyroscope

In the preceding articles it has been tacitly assumed that the gyroscope was symmetrical, that is, that $B = A$. This, however, is not always the case. In some gyroscopes of practical importance $B \neq A$. We now proceed to find the equations for steady or regular precession in this case.

Since for steady precession the quantities θ, $\dot{\phi}$ and $\dot{\psi}$ $(= \Omega)$ must all be constant, equations (19·1) become

$$\omega_x = \dot{\psi} \sin\theta \cos\phi,$$

$$\omega_y = \dot{\psi} \sin\theta \sin\phi,$$

$$\omega_z = \dot{\phi} + \dot{\psi} \cos\theta.$$

Hence

$$\frac{d\omega_x}{dt} = \dot{\phi}\dot{\psi} \sin\theta \sin\phi,$$

$$\frac{d\omega_y}{dt} = \dot{\phi}\dot{\psi} \sin\theta \cos\phi,$$

$$\frac{d\omega_z}{dt} = 0.$$

On substituting into (16·5) the above values of ω_x, ω_y, ω_z and their derivatives, we get

$$\left.\begin{aligned} (C-B)\,\dot{\psi}^2 \sin\theta \cos\theta \sin\phi + (A+C-B)\,\dot{\phi}\dot{\psi} \sin\theta \sin\phi &= M_x, \\ (C-A)\,\dot{\psi}^2 \sin\theta \cos\theta \cos\phi + (C-A+B)\,\dot{\phi}\dot{\psi} \sin\theta \cos\phi &= M_y, \\ (A-B)\,\dot{\psi}^2 \sin^2\theta \sin\phi \cos\phi &= M_z. \end{aligned}\right\} \tag{26·1}$$

The above equations (26·1) are referred to the axes of a trihedron which is fixed in the body and therefore spins with the same angular velocity as the body. Since it is rather absurd to use coordinate axes which may rotate thousands of times per minute, we choose a set of axes which rotate at a speed comparable to the speed of precession of the gyroscope. We therefore take the axis of spin, the axis of nodes, and a transverse axis perpendicular to each of these as the axes of the moving trihedron. The moving axes will therefore be OZ, ON and OB (Figs. 17 and 18). Then since angle

$XON = 90° - \phi$, the moments about ON and OB in terms of M_x and M_y are

$$\left.\begin{aligned} M_{ON} &= M_x \sin\phi + M_y \cos\phi, \\ M_{OB} &= M_x \cos\phi - M_y \sin\phi. \end{aligned}\right\} \qquad (26\cdot2)$$

Substituting into (26·2) the values of M_x and M_y from (26·1), we get

$$\begin{aligned} &[(C-B)\sin^2\phi + (C-A)\cos^2\phi]\,\dot\psi^2 \sin\theta\cos\theta \\ &\qquad + (B-A)(\cos^2\phi - \sin^2\phi)\,\dot\phi\dot\psi\sin\theta \\ &\qquad\qquad + C(\sin^2\phi + \cos^2\phi)\,\dot\phi\dot\psi\sin\theta = M_{ON}, \end{aligned}$$

$$\begin{aligned} (A-B)\,\dot\psi^2\sin\theta\cos\theta\sin\phi\cos\phi + 2(A-B)\,\dot\phi\dot\psi\sin\theta\sin\phi\cos\phi \\ = M_{OB}, \end{aligned}$$

$$(A-B)\,\dot\psi^2\sin^2\theta\sin\phi\cos\phi = M_z.$$

Now changing to double angles in ϕ by means of the relations

$$2\sin\phi\cos\phi = \sin 2\phi, \quad \cos^2\phi - \sin^2\phi = \cos 2\phi,$$

$$\sin^2\phi = \tfrac{1}{2}(1-\cos 2\phi), \quad \cos^2\phi = \tfrac{1}{2}(1+\cos 2\phi),$$

and then rearranging the results slightly, we get

$$\left.\begin{aligned} &\left[C\dot\phi + \left(C - \frac{A+B}{2}\right)\dot\psi\cos\theta\right]\dot\psi\sin\theta \\ &\qquad + \tfrac{1}{2}(B-A)(\dot\psi\cos\theta + 2\dot\phi)\,\dot\psi\sin\theta\cos 2\phi = M_{ON}, \\ &\qquad \tfrac{1}{2}(A-B)(\dot\psi\cos\theta + 2\dot\phi)\,\dot\psi\sin\theta\sin 2\phi = M_{OB}, \\ &\qquad\qquad \tfrac{1}{2}(A-B)\,\dot\psi^2\sin^2\theta\sin 2\phi = M_z. \end{aligned}\right\} \qquad (26\cdot3)$$

These equations (26·3) show that the moments about all the axes depend on the angle ϕ, that the moments are pulsating with period π, and that the pulsation frequency is therefore twice the rotational spin of the gyro. The moments about OB and OZ are zero twice during each rotation of the gyro, but the moment about ON is never zero unless

$$C\dot\phi + \left(C - \frac{A+B}{2}\right)\dot\psi\cos\theta = \tfrac{1}{2}(B-A)(\dot\psi\cos\theta + 2\dot\phi),$$

or

$$(C+A-B)\,\dot\phi = (B-C)\,\dot\psi\cos\theta,$$

which would rarely or never occur.

On rewriting the first two equations of (26·3) in the forms

$$\left[C+\left(C-\frac{A+B}{2}\right)\frac{\dot\psi}{\dot\phi}\cos\theta\right]\dot\phi\dot\psi\sin\theta$$
$$+\tfrac{1}{2}(B-A)\left(\frac{\dot\psi}{\dot\phi}\cos\theta+2\right)\dot\phi\dot\psi\sin\theta\cos 2\phi=M_{ON},$$
$$\tfrac{1}{2}(A-B)\left(\frac{\dot\psi}{\dot\phi}\cos\theta+2\right)\dot\phi\dot\psi\sin\theta\sin 2\phi=M_{OB},$$

we easily see that when the spin velocity $\dot\phi$ is very large and the precessional velocity $\dot\psi$ is consequently very small, the fraction $\dot\psi/\dot\phi$ is negligible. Furthermore, in this case the quantity

$$\tfrac{1}{2}(A-B)\,\dot\psi^2$$

is also very small and may be neglected. Hence, on dropping the negligible quantities just mentioned, we get

$$\left.\begin{aligned} C\dot\phi\dot\psi\sin\theta+(B-A)\,\dot\phi\dot\psi\sin\theta\cos 2\phi&=M_{ON},\\ (A-B)\,\dot\phi\dot\psi\sin\theta\sin 2\phi&=M_{OB},\end{aligned}\right\}\qquad(26{\cdot}4)$$

for the moments in a fast-spinning unsymmetrical gyroscope.

If we put $B=A$ in equations (26·3) and (26·4), we get equations (22·1) and (22·3), as we should.

27. Gyroscopic resistance

A rapidly spinning gyro is a stubborn affair. It resists any effort to turn it aside from its plane of rotation. To state it otherwise, it resists any force that tends to change the direction of its axis of spin.

In Art. 22 we found that an external moment which tended to increase the angle θ had to be applied in order to make the gyro precess at a constant rate. By Newton's third law of motion the gyro resisted the precession by an equal and opposite moment. This opposing moment is known as *the gyroscopic reaction moment*, the force of which is a reaction against the bearings and supports of the gyroscope. It is an inertial reaction ('gyroscopic inertia') analogous to centrifugal force and is the property that gives gyroscopes their practical usefulness. Henceforth in this book the term 'Gyroscopic Reaction Moment' will at times be abbreviated G.R.M.

As an illustration of this gyroscopic reaction moment, let us suppose that a gyroscope is rigidly installed on some moving body, such as a vehicle or a ship, and that its spin axis is horizontal. Then

if the moving body changes its direction of motion while the gyro is spinning at high speed, such a change in direction will change the direction of the spin axis by the same amount. This enforced change in direction of the spin axis is a forced precession about a vertical axis, and this precession will in turn automatically call forth a reaction moment against the bearings and supports of the gyroscope. This reaction moment is equal and opposite to the external moment that would be required to produce the forced precession. This instant gyroscopic action on the supports of the gyroscope tends to prevent the moving body from changing its direction of motion and thereby helps to steer it straight ahead.

Since the gyroscopic reaction moment is equal and opposite to the external applied moment, the equations giving the reaction moments can be obtained from the corresponding moment equations of Arts. 22, 25 and 26. Denoting gyroscopic reaction moment by K, we have, by definition,

$$K = -M, \quad K_x = -M_x, \quad K_y = -M_y, \quad \text{etc.}$$

Hence from equations (22·1)–(22·4) we get

$$K_y = -[C\dot{\phi} + (C-A)\,\Omega\cos\theta]\,\Omega\sin\theta, \tag{27·1}$$

$$K_y = -\left[C + (C-A)\frac{\Omega}{\dot{\phi}}\cos\theta\right]\Omega\dot{\phi}\sin\theta \tag{27·2}$$

$$= \left[C + (C-A)\frac{\Omega}{\dot{\phi}}\cos\theta\right](\dot{\boldsymbol{\phi}} \times \boldsymbol{\Omega}),$$

$$K_y = -C\dot{\phi}\Omega\sin\theta = C(\dot{\boldsymbol{\phi}} \times \boldsymbol{\Omega}), \tag{27·3}$$

$$K_y = -C\dot{\phi}\Omega = C(\dot{\boldsymbol{\phi}} \times \boldsymbol{\Omega}). \tag{27·4}$$

Likewise, from equations (25·2)–(25·4) we get

$$\left.\begin{aligned} K_x &= A\ddot{\psi}\sin\theta + 2A\dot{\theta}\dot{\psi}\cos\theta - C(\dot{\phi} + \dot{\psi}\cos\theta)\,\dot{\theta}, \\ K_y &= -\{A\ddot{\theta} + [C\dot{\phi} + (C-A)\,\dot{\psi}\cos\theta]\,\dot{\psi}\sin\theta\}, \\ K_z &= -C\frac{d}{dt}(\dot{\phi} + \dot{\psi}\cos\theta); \end{aligned}\right\} \tag{27·5}$$

$$\left.\begin{aligned} K_x &= -C\dot{\theta}\dot{\phi}, \\ K_y &= -C\dot{\phi}\dot{\psi}\sin\theta, \\ K_z &= -C\ddot{\phi}; \end{aligned}\right\} \tag{27·6}$$

$$\left.\begin{aligned} K_x &= -C\dot{\theta}\dot{\phi}, \\ K_y &= -C\dot{\phi}\dot{\psi}\sin\theta. \end{aligned}\right\} \tag{27·7}$$

If the quantities in the right-hand members of the above equations (27·1)–(27·7) are known, the reaction moments can be found.

The gyroscopic reaction moments for an unsymmetrical gyroscope are found from equations (26·3) and (26·4) to be

$$\left.\begin{aligned} K_{ON} &= -\left\{\left[C\dot{\phi}+\left(C-\frac{A+B}{2}\right)\dot{\psi}\cos\theta\right]\dot{\psi}\sin\theta \right. \\ &\qquad \left. +\tfrac{1}{2}(B-A)(\dot{\psi}\cos\theta+2\dot{\phi})\dot{\psi}\sin\theta\cos 2\phi\right\}, \\ K_{OB} &= -\left[\tfrac{1}{2}(A-B)(\dot{\psi}\cos\theta+2\dot{\phi})\dot{\psi}\sin\theta\sin 2\phi\right], \\ K_z &= -\left[\tfrac{1}{2}(A-B)\dot{\psi}^2\sin^2\theta\sin 2\phi\right], \end{aligned}\right\} \tag{27·8}$$

and

$$\left.\begin{aligned} K_{ON} &= -[C\dot{\phi}\dot{\psi}\sin\theta+(B-A)\dot{\phi}\dot{\psi}\sin\theta\cos 2\phi], \\ K_{OB} &= -[(A-B)\dot{\phi}\dot{\psi}\sin\theta\sin 2\phi]. \end{aligned}\right\} \tag{27·9}$$

We see at once from these equations that the G.R.M.'s in an unsymmetrical gyroscope are pulsating with a frequency twice the rotational speed of the gyro. If these pulsations happen to be nearly the same as the natural vibration frequency of some part or parts of the machine to which the gyroscope is attached, resonance will take place and may cause serious damage.

Because of the very great importance of the principle of gyroscopic reaction, we state again by way of emphasis that if a gyro is forced by any means to precess with an angular velocity Ω about any axis, it will immediately exert on its bearings or supports a reactionary moment equal and opposite to the external moment that would be required to produce the precession. The magnitude of this gyroscopic reaction moment is given by the formula

$$K = C\Omega\dot{\phi} \tag{27·10}$$

when the axis of precession is at right angles to the axis of spin.

The fundamental relations between external moment and precession, and between precession and gyroscopic reaction moment, have been succinctly stated by Grammel.*

To a moment $\mathbf{M}$ *the gyro responds by a precession* $\boldsymbol{\Omega}$; *a precession* $\boldsymbol{\Omega}$ *induces in the gyro a reactionary moment* $\mathbf{K}$.

From the standpoint of practical usefulness equation (27·2) and its special case (27·4) are the most important formulas in gyroscopic

* *Der Kreisel*, p. 72, 1920 edition.

theory. Particular attention is called to (27·4). Reference to Fig. 20 will show that a moment about OY tends to change the direction of the axis of spin and therefore to deflect the gyro from its plane of rotation. When a heavy gyro is spinning at high speed, the right-hand member of (27·4) is large and therefore the reaction moment $\mathbf{K}$ is also large. This means that the gyro offers great resistance to being deflected from its plane of rotation. The higher the spin velocity and the greater the moment of inertia of the gyro, the greater is the deviation resistance.

A glance at equation (27·4), or (27·10), shows that *there can be no gyroscopic resistance unless there is precession,* for when Ω is zero, $\mathbf{K}$ is also zero.

28. Direction of the gyroscopic reaction moment

We come now to a consideration of the all-important question of the *direction* of the gyroscopic reaction moment, for in the applications of the gyroscope it is even more important to know the direction of the reaction than to know its magnitude. The vector equation $\mathbf{K}=C(\dot{\boldsymbol{\phi}}\times\boldsymbol{\Omega})$ shows, according to the definition of the vector product, that the reaction $\mathbf{K}$ tends to turn the spin vector into the direction of the precession vector. This fact is also indicated in Fig. 21. Hence we can state the following rule for determining the direction of the gyroscopic reaction moment:

The gyroscopic reaction moment (G.R.M.) *tends to turn the positive end of the spin axis toward the positive end of the precession axis,* the positive end being determined in each case by the right-handed-screw rule.

Note that the dotted curved arrows in Fig. 21 are drawn from one known vector to another known vector in each case.

29. Motion of a free gyroscope

A free gyroscope is one on which no external forces act, or one suspended at its center of gravity and acted on by no other force than its own weight. In either case the external moment about its point of suspension is zero. Then, since $\mathbf{M}=0$, equation (15·1) becomes

$$\frac{d\mathbf{H}}{dt}=0.$$

Consequently, $\mathbf{H}=\text{a constant.}$

Since equation (15·1) refers the motion to a set of *fixed* axes, and since **H** is constant in both magnitude and direction, it is evident that **H** has a *fixed direction in space*. The fixed line on which **H** lies is called the *invariable line*, and any plane perpendicular to it is called an invariable plane.*

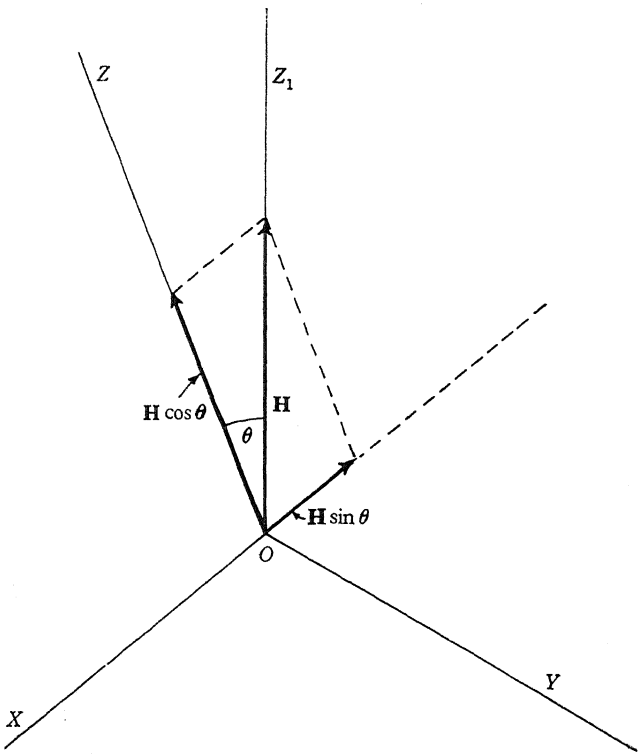


Fig. 24

To find the motion of a free gyroscope we therefore take the invariable line as the fixed vertical axis. The x-axis is taken in the ZOH-plane, and the y-axis is taken perpendicular to this plane, as

* The solar system is not acted on appreciably by any external forces. Hence its moment of momentum is constant. It therefore has an invariable line. The plane through the center of the sun and perpendicular to the invariable line is the invariable plane of the system. It is inclined to the plane of the ecliptic by about 2°.

indicated in Fig. 24. Then since the external moments are all zero, equations (25·2) become

$$\left.\begin{aligned} -A\ddot{\psi}\sin\theta - 2A\dot{\theta}\dot{\psi}\cos\theta + C(\dot{\phi}+\dot{\psi}\cos\theta)\,\dot{\theta} &= 0,\\ A\ddot{\theta} + [C\dot{\phi} + (C-A)\,\dot{\psi}\cos\theta]\,\dot{\psi}\sin\theta &= 0,\\ C\frac{d}{dt}(\dot{\phi}+\dot{\psi}\cos\theta) &= 0. \end{aligned}\right\} \tag{29·1}$$

Our problem now is to integrate these equations.

From Fig. 24 we have

$$H_x = -H\sin\theta, \quad H_y = 0, \quad H_z = H\cos\theta.$$

Now integrating the third of equations (29·1), we get

$$C(\dot{\phi}+\dot{\psi}\cos\theta) = \text{a constant}.$$

But $$H_z = H\cos\theta = C(\dot{\phi}+\dot{\psi}\cos\theta) = \text{a constant}.$$

Therefore, since H and $H\cos\theta$ are both constant, $\cos\theta$ must also be constant. Hence

$$\theta = \text{a constant},$$

and $$\dot{\theta} = 0, \quad \ddot{\theta} = 0.$$

Substituting these values for $\dot{\theta}$ and $\ddot{\theta}$ into the first two of equations (29·1), we get

$$\left.\begin{aligned} \ddot{\psi} &= 0,\\ C\dot{\phi} + (C-A)\,\dot{\psi}\cos\theta &= 0. \end{aligned}\right\} \tag{29·2}$$

From the first of these equations we get

$$\dot{\psi} = \text{a constant} = \Omega, \quad \text{say}.$$

Substituting this for $\dot{\psi}$ into the second of equations (29·2), we get

$$C\dot{\phi} + (C-A)\,\Omega\cos\theta = 0.$$

Hence $$\dot{\phi} = -\frac{C-A}{C}\,\Omega\cos\theta, \quad \text{a constant}. \tag{29·3}$$

We have now shown that θ, $\dot{\psi}$ and $\dot{\phi}$ are all constant for a free gyroscope. The motion is therefore that of steady precession. It will be noted that equation (29·3) is the same as (24·1), and the precession is therefore a free precession.

We now show that in this motion the instantaneous axis of rotation makes a constant angle with the invariable line. Since the rotation about the instantaneous axis is the geometric sum of the spin velocity and the precession velocity, the instantaneous axis is

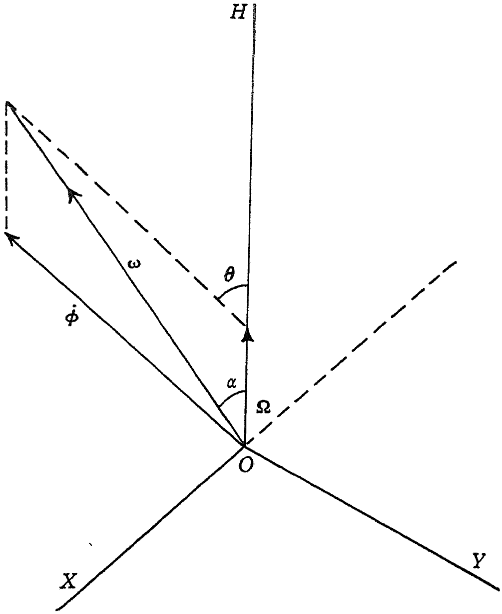


Fig. 25

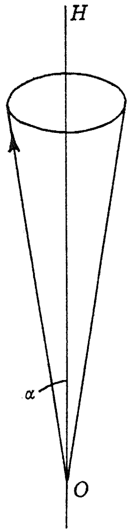


Fig. 26

represented by the vector $\boldsymbol{\omega}$ in Fig. 25. If α denotes the angle which $\boldsymbol{\omega}$ makes with the invariable line we have, by the law of sines,

$$\frac{\dot{\phi}}{\sin\alpha}=\frac{\omega}{\sin\theta},$$

from which
$$\sin\alpha=\frac{\dot{\phi}\sin\theta}{\omega}. \tag{29·4}$$

Since $\dot{\phi}$, θ and ω are all constants, $\sin\alpha$ is constant and therefore

$$\alpha=\text{a constant}.$$

The instantaneous axis thus describes a cone of semivertical angle α about the invariable line as an axis, as shown in Fig. 26. Equation (29·4) shows that α depends on the velocity of spin, the velocity of precession, and the angle θ.

30. Stability of the motion of a free gyroscope

We now investigate the stability of the motion of a gyroscope on which no external couples are acting. Consider a gyroscope that is free to move in any manner about its center of gravity as a fixed point. To simplify the problem we assume that the axis of spin is horizontal, that the gyro is spinning with angular velocity $\dot{\phi}$, and that the spin axis may or may not be precessing with angular velocity $\dot{\psi}$. Assume now that the axle of the gyro is struck a sharp impulsive blow I, directed vertically downward as indicated in Fig. 27. Our problem is to determine the effect of this blow on the subsequent motion of the gyroscope.

The immediate effect of the blow is to give the spin axis an angular velocity $\dot{\theta}_0$ in the azimuthal plane. Hence the initial conditions of the problem are

$$\theta = \tfrac{1}{2}\pi, \quad \dot{\theta} = \dot{\theta}_0, \quad \dot{\psi} = \dot{\psi}_0, \quad \dot{\phi} = \dot{\phi}_0, \quad \psi = 0, \quad \text{when } t = 0.$$

On putting $\theta = \frac{1}{2}\pi$ and $M_x = M_y = M_z = 0$ in equations (25·2), these equations become

$$\left.\begin{aligned} -A\ddot{\psi} + C\dot{\theta}\dot{\phi} &= 0, \\ A\ddot{\theta} + C\dot{\phi}\dot{\psi} &= 0, \\ C\ddot{\phi} &= 0. \end{aligned}\right\} \qquad (30·1)$$

From the third of equations (30·1) we get

$$C\dot{\phi} = \text{constant} = C\omega, \quad \text{say.}$$

Then
$$\dot{\phi} = \omega = \dot{\phi}_0 \quad \text{and} \quad \phi = \omega t + \phi_0.$$

Substituting this value of $\dot{\phi}$ into the first two of equations (30·1), we get

$$\left.\begin{aligned} -A\ddot{\psi} + C\omega\dot{\theta} &= 0, \\ A\ddot{\theta} + C\omega\dot{\psi} &= 0. \end{aligned}\right\} \qquad (30·2)$$

Integration of the first of these equations gives

$$-A\dot{\psi} + C\omega\theta = C_1.$$

To find C_1, put $\dot{\psi} = \dot{\psi}_0$ and $\theta = \frac{1}{2}\pi$. Then

$$C_1 = C\omega\frac{\pi}{2} - A\dot{\psi}_0.$$

The above equation now becomes

$$-A\dot{\psi}+C\omega\theta=C\omega\frac{\pi}{2}-A\dot{\psi}_0,$$

or

$$\dot{\psi}=\dot{\psi}_0+\frac{C\omega}{A}(\theta-\tfrac{1}{2}\pi). \tag{30·3}$$

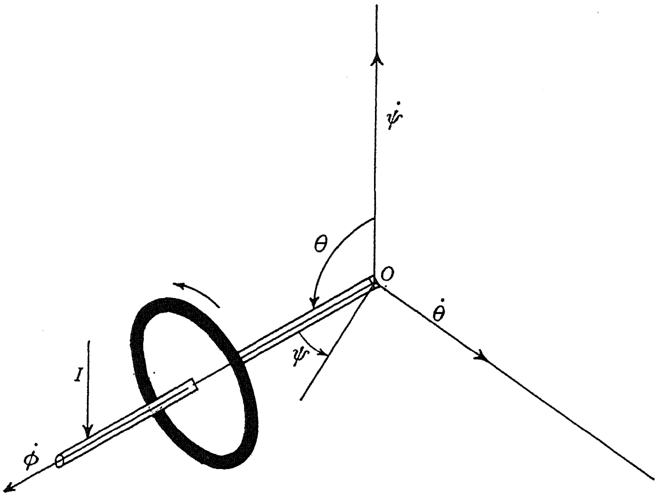


Fig. 27

Substituting this value of $\dot{\psi}$ into the second of equations (30·2), we obtain

$$A\ddot{\theta}+C\omega\left[\dot{\psi}_0+\frac{C\omega}{A}(\theta-\tfrac{1}{2}\pi)\right]=0,$$

or

$$\ddot{\theta}+\left(\frac{C\omega}{A}\right)^2\theta=\left(\frac{C\omega}{A}\right)\frac{\pi}{2}-\frac{C\omega}{A}\dot{\psi}_0. \tag{30·4}$$

Solving (30·4) by the method of undetermined coefficients, we have

$$r^2+\left(\frac{C\omega}{A}\right)^2=0,$$

whence

$$r=\pm i\left(\frac{C\omega}{A}\right).$$

Then the complementary function is

$$\theta_c = C_2 \sin\left(\frac{C\omega}{A}\right) t + C_3 \cos\left(\frac{C\omega}{A}\right) t.$$

Since the right-hand member of (30·4) is a constant, we find a particular integral by putting

$$\theta_p = P, \quad \text{a constant.}$$

Then $$\dot{\theta} = 0, \quad \ddot{\theta} = 0.$$

Substituting these values of θ and $\ddot{\theta}$ into (30·4), we get

$$\left(\frac{C\omega}{A}\right)^2 P = \left(\frac{C\omega}{A}\right)^2 \frac{\pi}{2} - \frac{C\omega}{A}\dot{\psi}_0,$$

whence $$P = \frac{\pi}{2} - \frac{A}{C\omega}\dot{\psi}_0.$$

The complete solution of (30·4) is therefore

$$\theta = C_2 \sin\left(\frac{C\omega}{A}\right) t + C_3 \cos\left(\frac{C\omega}{A}\right) t + \frac{\pi}{2} - \frac{A}{C\omega}\dot{\psi}_0.$$

Then $$\dot{\theta} = C_2 \frac{C\omega}{A} \cos\left(\frac{C\omega}{A}\right) t - C_3 \frac{C\omega}{A} \sin\left(\frac{C\omega}{A}\right) t.$$

To find C_2 and C_3 we utilize the initial values $\theta = \frac{1}{2}\pi$, $\dot{\theta} = \dot{\theta}_0$, when $t = 0$. Then we find $C_2 = A\dot{\theta}_0/C\omega$, $C_3 = A\dot{\psi}_0/C\omega$. The value of θ is therefore

$$\theta = \frac{\pi}{2} - \frac{A\dot{\psi}_0}{C\omega} + \frac{A}{C\omega}\left(\dot{\theta}_0 \sin\frac{C\omega}{A} t + \dot{\psi}_0 \cos\frac{C\omega}{A} t\right)$$

$$= \frac{\pi}{2} - \frac{A\dot{\psi}_0}{C\omega} + \frac{A}{C\omega}\sqrt{(\dot{\theta}_0^2 + \dot{\psi}_0^2)} \sin\left(\frac{C\omega t}{A} + \beta\right), \tag{30·5}$$

where $\beta = \tan^{-1}\dot{\psi}_0/\dot{\theta}_0$.

Now substituting this value of θ into (30·3), we get

$$\dot{\psi} = \dot{\psi}_0 + \frac{C\omega}{A}\left(-\frac{A\dot{\psi}_0}{C\omega} + \frac{A}{C\omega}\sqrt{(\dot{\theta}_0^2 + \dot{\psi}_0^2)} \sin\left(\frac{C\omega}{A} t + \beta\right)\right)$$

$$= \sqrt{(\dot{\theta}_0^2 + \dot{\psi}_0^2)} \sin\left(\frac{C\omega}{A} t + \beta\right). \tag{30·6}$$

Integration of this gives

$$\psi = -\frac{A}{C\omega}\sqrt{(\dot{\theta}_0^2 + \dot{\psi}_0^2)} \cos\left(\frac{C\omega}{A} t + \beta\right) + C_4.$$

To find C_4 we utilize the initial condition $\psi=0$ when $t=0$. This gives

$$C_4=\frac{A}{C\omega}\sqrt{(\dot\theta_0^2+\dot\psi_0^2)}\cos\beta=\frac{A}{C\omega}\sqrt{(\dot\theta_0^2+\dot\psi_0^2)}\frac{\dot\theta_0}{\sqrt{(\dot\theta_0+\dot\psi_0^2)}}=\frac{A\dot\theta_0}{C\omega}.$$

Then ψ becomes

$$\psi=\frac{A\dot\theta_0}{C\omega}-\frac{A}{C\omega}\sqrt{(\dot\theta_0^2+\dot\psi_0^2)}\cos\left(\frac{C\omega}{A}t+\beta\right). \tag{30·7}$$

The value of θ in (30·5) can be written in the equivalent form

$$\theta=\frac{\pi}{2}-\frac{A\dot\psi_0}{C\omega}-\frac{A}{C\omega}\sqrt{(\dot\theta_0^2+\dot\psi_0^2)}\cos\left(\frac{C\omega t}{A}+\beta+\frac{\pi}{2}\right), \tag{30·8}$$

which shows that the periodic parts of θ and ψ have the same period but are out of phase by a quarter period.

The final solution of equations (30·1) is therefore:

$$\left.\begin{aligned}
&\phi=\omega t+\phi_0,\\
&\theta=\frac{\pi}{2}-\frac{A\dot\psi_0}{C\omega}+\frac{A}{C\omega}\sqrt{(\dot\theta_0^2+\dot\psi_0^2)}\sin\left(\frac{C\omega t}{A}+\beta\right),\\
&\psi=\frac{A\dot\theta_0}{C\omega}-\frac{A}{C\omega}\sqrt{(\dot\theta_0^2+\dot\psi_0^2)}\cos\left(\frac{C\omega t}{A}+\beta\right),\\
&\beta=\tan^{-1}\dot\psi_0/\dot\theta_0.
\end{aligned}\right\} \tag{30·9}$$

These equations show that after the impulsive blow the gyro continued to spin with the same angular velocity as before, but that the axis of spin was displaced from the position $\theta=\frac{1}{2}\pi$, $\psi=0$ to the mean position $\theta=\frac{1}{2}\pi-A\dot\psi_0/C\omega$, $\psi=A\dot\theta_0/C\omega$, and that it thereafter oscillated about this mean position with simple harmonic motion both vertically and horizontally, the vertical and horizontal oscillations being out of phase by a quarter of a period. It is to be noted here that if the gyro axis was precessing at the instant of the blow, the progressive precession ceased after the axis reached the position for which $\psi=A\dot\theta_0/C\omega$; for equation (30·6) shows that the only precession is periodic. We note further that the displaced position of the axis depends on the intensity of the blow ($\dot\theta_0$) and on the angular velocity of spin—the greater the velocity of spin, the smaller the displacement.

The amplitude of the harmonic oscillations is $(A/C\omega)\sqrt{(\dot\theta^2+\dot\psi^2)}$ and their frequency is $C\omega/2\pi A$, thus showing that the higher

the spin velocity, the smaller the amplitude and the higher the frequency.

The preceding investigation shows that the motion of a free symmetrical gyroscope is stable. Moreover, if A and C be interchanged, the motion is still stable. Hence the motion is stable whether the axis of spin is about the A-principal axis of inertia or about the C-principal axis.

To investigate the stability of a free unsymmetrical gyroscope, we utilize equations (16·5) and (19·1). We first consider the case in which the spin axis coincides with OZ and assume that the axle is struck a sharp blow in the ZOZ_1-plane and at right angles to the axle. The immediate effect of the blow is to impart an angular velocity $\dot{\theta}_0$ to the axle (see Fig. 20).

Turning now to equations (19·1), we notice that ω_x and ω_y are very small in comparison with ω_z, because $\dot{\theta}$ and $\dot{\psi}$ are very small in comparison with the high spin velocity $\dot{\phi}$. Hence we shall neglect the product $\omega_x\omega_y$ in the third equation of (16·5). Then, since $M_x = M_y = M_z = 0$ in a free gyroscope, equations (16·5) become

$$\left.\begin{aligned} A\frac{d\omega_x}{dt} + (C-B)\,\omega_y\omega_z &= 0,\\ B\frac{d\omega_y}{dt} + (A-C)\,\omega_x\omega_z &= 0,\\ C\frac{d\omega_z}{dt} \qquad\qquad\qquad &= 0. \end{aligned}\right\} \tag{30·10}$$

From the third of these equations we get

$$\omega_z = \text{a constant} = \Omega, \quad \text{say.}$$

Substituting this value of ω_z into the first and second of equations (30·10), we have

$$\left.\begin{aligned} A\frac{d\omega_x}{dt} + (C-B)\,\Omega\omega_y &= 0,\\ B\frac{d\omega_y}{dt} + (A-C)\,\Omega\omega_x &= 0. \end{aligned}\right\} \tag{30·11}$$

Now differentiating the first equation in (30·11) and then substituting the resulting value of $d\omega_y/dt$ into the second equation, we get

$$AB\frac{d^2\omega_x}{dt^2} + (A-C)(B-C)\,\Omega^2\omega_x = 0.$$

Here the auxiliary equation is

$$ABr^2+(A-C)(B-C)\Omega^2=0,$$

from which $$r=\pm i\Omega\sqrt{\frac{(A-C)(B-C)}{AB}}.$$

The period of oscillation resulting from the blow is therefore

$$T=\frac{2\pi}{\Omega}\sqrt{\frac{AB}{(A-C)(B-C)}}.$$

This period is real either for $A>B>C$ or for $C>B>A$, and the gyroscope is therefore stable whether the axis of spin is along the A-principal axis or the C-principal axis of the gyro.

We next consider the case in which the axis of spin coincides with the y-axis and therefore the B-principal axis of the gyro. In this case ω_x and ω_z are very small and their product negligible, while ω_y is large. Hence equations (16·5) become

$$\left.\begin{aligned} A\frac{d\omega_x}{dt}+(C-B)\,\omega_y\omega_z&=0,\\ B\frac{d\omega_y}{dt}\qquad\qquad\quad&=0,\\ C\frac{d\omega_z}{dt}+(B-A)\,\omega_x\omega_y&=0.\end{aligned}\right\}\qquad(30\cdot12)$$

From the second equation of (30·12), we get

$$\omega_y=\text{a constant}=\Omega,\quad\text{say.}$$

On substituting this into the first and third of equations (30·12), we get

$$\left.\begin{aligned} A\frac{d\omega_x}{dt}+(C-B)\,\Omega\omega_z&=0,\\ C\frac{d\omega_z}{dt}+(B-A)\,\Omega\omega_x&=0.\end{aligned}\right\}\qquad(30\cdot13)$$

Differentiating the first equation and substituting the resulting value of $d\omega_z/dt$ into the second, we find

$$AC\frac{d^2\omega_x}{dt^2}+(B-A)(B-C)\,\Omega^2\omega_x=0.$$

Here the auxiliary equation is

$$ACr^2 - \Omega^2(A-B)(B-C) = 0,$$

from which
$$r = \pm\Omega\sqrt{\frac{(A-B)(B-C)}{AC}}.$$

The values of r are real, since the factors under the radical are both positive for $A > B > C$ or $C > B > A$. Hence ω_x is not periodic and the spin axis does not oscillate after the blow. The gyroscope is therefore unstable when spinning about the intermediate or B-principal axis of inertia.

We may therefore state that a free unsymmetrical gyroscope is stable when spinning about either its greatest or least principal axis of inertia, but is unstable when spinning about its intermediate principal axis.

31. Motion of the spin axis in unsteady precession

We next consider the motion of a gyroscope which is spinning at high speed and also precessing under the action of a constant external moment. Let N denote the external moment and let it be applied about the axis of nodes (y-axis). Then

$$M_y = N \quad \text{and} \quad M_x = M_z = 0.$$

With $\theta = 90°$, as in Art. 30, equations (25·2) now become

$$\left.\begin{aligned} -A\ddot{\psi} + C\dot{\phi}\dot{\theta} &= 0, \\ A\ddot{\theta} + C\dot{\phi}\dot{\psi} &= N, \\ C\ddot{\phi} &= 0. \end{aligned}\right\} \qquad (31{\cdot}1)$$

These equations are to be solved subject to the initial conditions:

$$\theta = \tfrac{1}{2}\pi, \quad \dot{\theta} = 0, \quad \dot{\psi} = \dot{\psi}_0, \quad \text{when } t = 0.$$

From the third equation of (31·1) we get

$$C\dot{\phi} = \text{a constant} = C\omega, \quad \text{say}.$$

Then
$$\dot{\phi} = \omega \quad \text{and} \quad \phi = \omega t + \phi_0.$$

Substituting this value of $\dot{\phi}$ into the first two of equations (31·1), we have

$$\left.\begin{aligned} -A\ddot{\psi} + C\omega\dot{\theta} &= 0, \\ A\ddot{\theta} + C\omega\dot{\psi} &= N. \end{aligned}\right\} \qquad (31{\cdot}2)$$

Integration of the first of these equations gives

$$-A\dot{\psi}+C\omega\theta=C_1. \tag{31·3}$$

To find C_1 we put $\theta=\frac{1}{2}\pi$ and $\dot{\psi}=\dot{\psi}_0$. Then

$$C_1=C\omega\frac{\pi}{2}-A\dot{\psi}_0,$$

and (31·3) becomes

$$-A\dot{\psi}+C\omega\theta=C\omega\frac{\pi}{2}-A\dot{\psi}_0,$$

from which

$$\dot{\psi}=\frac{C\omega}{A}\left(\theta-\frac{\pi}{2}\right)+\dot{\psi}_0. \tag{31·4}$$

When this value of $\dot{\psi}$ is substituted into the second of equations (31·2), we have

$$A\ddot{\theta}+C\omega\left[\frac{C\omega}{A}\left(\theta-\frac{\pi}{2}\right)+\dot{\psi}_0\right]=N,$$

which becomes

$$\ddot{\theta}+\left(\frac{C\omega}{A}\right)^2\theta=\frac{N}{A}-\frac{C\omega}{A}\dot{\psi}_0+\left(\frac{C\omega}{A}\right)^2\frac{\pi}{2}. \tag{31·5}$$

Solving this equation by the method of undetermined coefficients, as in Art. 30, we have

$$\theta_c=C_2\sin\frac{C\omega}{A}t+C_3\cos\frac{C\omega}{A}t.$$

To find a particular integral we put $\theta_p=P$ and substitute it and its second derivative into (31·5). We thus get

$$\left(\frac{C\omega}{A}\right)^2P=\frac{N}{A}+\left(\frac{C\omega}{A}\right)^2\frac{\pi}{2}-\frac{C\omega\dot{\psi}_0}{A},$$

whence

$$P=\frac{NA}{(C\omega)^2}+\frac{\pi}{2}-\frac{A}{C\omega}\dot{\psi}_0.$$

The value of θ is therefore

$$\theta=C_2\sin\frac{C\omega}{A}t+C_3\cos\frac{C\omega}{A}t+\frac{NA}{(C\omega)^2}+\frac{\pi}{2}-\frac{A}{C\omega}\dot{\psi}_0.$$

Then

$$\dot{\theta}=C_2\frac{C\omega}{A}\cos\frac{C\omega}{A}t-C_3\frac{C\omega}{A}\sin\frac{C\omega}{A}t.$$

To find C_2 and C_3 we put $\theta=\frac{1}{2}\pi$, $\dot\theta=0$ when $t=0$. Thus

$$\frac{\pi}{2}=C_2\frac{NA}{(C\omega)^2}+\frac{\pi}{2}-\frac{A}{C\omega}\dot\psi_0,$$

whence

$$C_3=\frac{A}{C\omega}\dot\psi_0-\frac{NA}{(C\omega)^2}.$$

And

$$0=C_2\frac{C\omega}{A}\quad\text{or}\quad C_2=0.$$

Then θ and $\dot\theta$ become

$$\left.\begin{aligned}\theta&=\frac{A}{C\omega}\left(\frac{N}{C\omega}-\dot\psi_0\right)\left(1-\cos\frac{C\omega}{A}t\right)+\frac{\pi}{2},\\ \dot\theta&=\left(\frac{N}{C\omega}-\dot\psi_0\right)\sin\frac{C\omega}{A}t.\end{aligned}\right\}\tag{31·6}$$

Now substituting this value of θ into equation (31·4), we get

$$\dot\psi=\left(\frac{N}{C\omega}-\dot\psi_0\right)\left(1-\cos\frac{C\omega}{A}t\right)+\dot\psi_0.$$

Integration of this gives

$$\psi=\left(\frac{N}{C\omega}-\dot\psi_0\right)\left(t-\frac{A}{C\omega}\sin\frac{C\omega}{A}t\right)+\dot\psi_0 t+C_4.$$

Putting $\psi=0$, $t=0$, we get $C_4=0$. Hence

$$\psi=\frac{N}{C\omega}t+\frac{A}{C\omega}\left(\dot\psi_0-\frac{N}{C\omega}\right)\sin\frac{C\omega}{A}t.\tag{31·7}$$

The final solutions of equations (31·1) are therefore

$$\left.\begin{aligned}&\dot\phi=\omega=\text{a constant}\quad\text{or}\quad\phi=\omega t,\\ &\theta=\frac{\pi}{2}-\frac{A}{C\omega}\left(\dot\psi_0-\frac{N}{C\omega}\right)+\frac{A}{C\omega}\left(\dot\psi_0-\frac{N}{C\omega}\right)\cos\frac{C\omega t}{A},\\ &\psi=\frac{N}{C\omega}t+\frac{A}{C\omega}\left(\dot\psi_0-\frac{N}{C\omega}\right)\sin\frac{C\omega}{A}t.\end{aligned}\right\}\tag{31·8}$$

These equations show that the velocity of spin remains constant, that the precession continues at a uniform rate except for a periodic disturbance which alternately speeds it up and slows it down, and

that the axis of spin oscillates in the azimuthal plane. The mean position of the spin axis is not in the plane $\theta=\frac{1}{2}\pi$ but lies in the conical surface

$$\theta=\frac{\pi}{2}+\frac{A}{C\omega}\left(\frac{N}{C\omega}-\dot{\psi}_0\right).$$

If $\dot{\psi}_0=N/C\omega$, or $N=C\omega\dot{\psi}_0$, all vibration of the spin axis disappears and it precesses at a constant rate in the plane $\theta=\frac{1}{2}\pi$. Since the equation $N=C\omega\dot{\psi}$ is the same as (22·4), the motion is steady precession. There is therefore no vibration of the axis when the precession is steady.

On writing the equation for θ in the form

$$\theta=\frac{\pi}{2}-\frac{A}{C\omega}\left(\dot{\psi}_0-\frac{N}{C\omega}\right)+\frac{A}{C\omega}\left(\dot{\psi}_0-\frac{N}{C\omega}\right)\sin\left(\frac{C\omega t}{A}+\frac{\pi}{2}\right),$$

and comparing this result with the value for ψ, we note that the oscillations have the same periods and amplitudes in both cases but that they differ in phase by a quarter of a period.

Note. This book deals with the ideal gyroscope, free from friction and all imperfections of construction. Although the spin velocity of the rotor is kept constant by an electric motor, the gimbal bearings are not absolutely perfect, and it is not possible to keep the center of gravity of the rotor and gimbal assembly at the exact point of intersection of the gimbal axes. These departures from the ideal state cause the spin axis to depart slightly and slowly from its initial orientation. This slow change in orientation of the spin axis is called *drift*. Designers and manufacturers of gyroscopes make every effort to reduce drift to a minimum.

CHAPTER IV

Theory of the Gyroscope (Continued) Motion of a Gyroscope under the Action of Gravity. The Top

32. The differential equations of motion

Many gyroscopes are suspended in such a manner that the point of suspension is not at the center of gravity. In such cases the weight of the gyroscope has a moment about the point of suspension. The ordinary top is a gyroscope of this type. Fig. 28 represents a gyroscope suspended in the manner just mentioned.

To study the motion of a gyroscope under the action of gravity, we take a set of moving axes $O-XYZ$ with the z-axis along the axis of spin, the x-axis in the ZOZ_1 plane, and the y-axis perpendicular to this plane, as indicated in Fig. 28. Let W denote the weight of the gyroscope, or top, and let l denote the distance OG. Then since the y-axis is perpendicular to the plane ZOZ_1, the moment of the weight is about the y-axis. Hence we have $M_x=0$, $M_y=Wl\sin\theta$, $M_z=0$. Substituting these values into equations (25·2), we get

$$\left.\begin{aligned} -A\ddot{\psi}\sin\theta-2A\dot{\theta}\dot{\psi}\cos\theta+C(\dot{\phi}+\dot{\psi}\cos\theta)\,\dot{\theta}&=0,\\ A\ddot{\theta}+[C\dot{\phi}+(C-A)\,\dot{\psi}\cos\theta]\,\dot{\psi}\sin\theta&=Wl\sin\theta,\\ C\frac{d}{dt}(\dot{\phi}+\dot{\psi}\cos\theta)&=0,\end{aligned}\right\}\qquad(32\cdot1)$$

as the differential equations of motion. Before attempting to find the general solutions of these equations, we shall consider particular cases and aspects of them.

33. Forced steady precession

For steady precession we must have (Art. 22) $\theta=\text{constant}$, $\dot{\phi}=\text{constant}$ and $\dot{\psi}=\text{constant}$. Then $\dot{\theta}=\ddot{\theta}=0$, $\ddot{\psi}=0$ and

$$\dot{\phi}+\dot{\psi}\cos\theta=\text{constant}=\omega,\quad\text{say.}$$

When these values are substituted into (32·1), these equations become

$$(C\omega - A\dot{\psi}\cos\theta)\,\dot{\psi} = Wl. \tag{33·1}$$

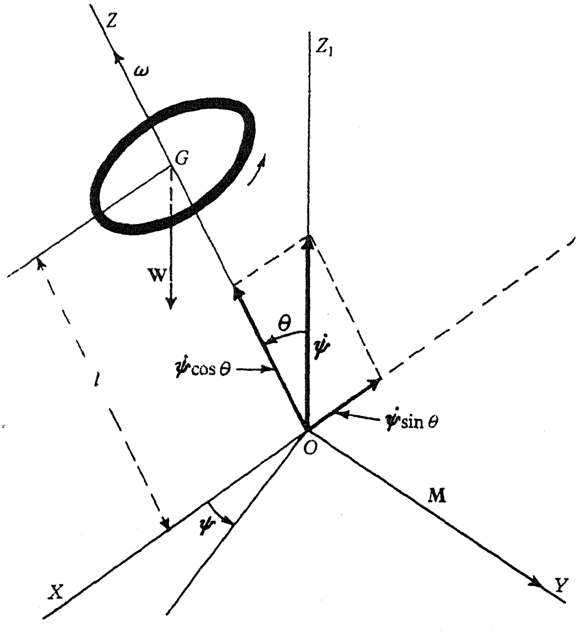


Fig. 28

This is the condition for the *maintenance* of steady precession under the action of gravity. Writing (33·1) in the form

$$\left(C - A\frac{\dot{\psi}}{\omega}\cos\theta\right)\omega\dot{\psi} = Wl,$$

we see that for a high velocity of spin the fraction $\dot{\psi}/\omega$ is negligible and that the equation thus reduces to

$$C\omega\dot{\psi} = Wl,$$

or

$$\dot{\psi} = \frac{Wl}{C\omega}, \tag{33·2}$$

regardless of the magnitude of θ. This is the same equation as (22·4). It shows that the velocity of precession varies directly as the distance l ($=OG$ in Fig. 28).

On putting $C=Mk^2=\frac{W}{g}k^2$, we may write (33·2) in the equivalent form

$$\dot{\psi}=\frac{Wl}{\frac{W}{g}k^2\omega}=\frac{gl}{k^2\omega}, \tag{33·3}$$

where k denotes the radius of gyration of the gyro about its geometric axis. This equation shows that the rate of precession, for a given rate of spin, is determined by the *shape* of the gyro. When the gyro is in the form of a flat circular disk, or flywheel, the rate of precession is very slow; whereas if the gyro has the form of a slender body such as a rod, the rate of precession is very fast.

For $\theta=90°$, equation (33·1) reduces to (33·2) regardless of the ratio $\dot{\psi}/\omega$.

Since equation (33·1) is a quadratic in $\dot{\psi}$, there are *two* rates of steady precession. These are found by solving (33·1) to be

$$\dot{\psi}=\frac{C\omega\pm\sqrt{(C^2\omega^2-4AWl\cos\theta)}}{2A\cos\theta}. \tag{33·4}$$

If $0°<\theta<90°$, the radicand is less than $C\omega$ and both values of $\dot{\psi}$ are positive; whereas if $90°<\theta<180°$, the radicand is greater than $C\omega$, thereby making one value of $\dot{\psi}$ positive and the other negative. It is to be noted that when $\dot{\psi}$ is positive the precession has the same direction as the spin of the gyro, whereas if $\dot{\psi}$ is negative the direction of precession is opposite to that of the spin.

The roots of (33·4) are equal when the radicand is zero; that is, when

$$C^2\omega^2=4AWl\cos\theta,$$

or

$$\omega=\frac{2\sqrt{(AWl\cos\theta)}}{C}\quad(\theta\neq\tfrac{1}{2}\pi). \tag{33·5}$$

Any smaller value of ω will make the radicand of (33·4) negative and thereby give imaginary values for $\dot{\psi}$. Hence (33·5) gives the minimum spin for precession. In this case the value of $\dot{\psi}$ is, by (33·4) and (33·5),

$$\dot{\psi}=\frac{C\omega}{2A\cos\theta}=\sqrt{\frac{Wl}{A\cos\theta}}\quad(\theta\neq\tfrac{1}{2}\pi). \tag{33·6}$$

Solving (33·1) for $\cos\theta$, we get

$$\cos\theta = \frac{C\omega\dot{\psi} - Wl}{A\dot{\psi}^2}. \tag{33·7}$$

This equation shows that in steady precession an increase in C causes a decrease in θ, and an increase in A causes an increase in θ. This means that the geometric axis of a spinning disk will make a smaller angle with the vertical than will the axis of a slender rod spinning about its geometric axis. This fact can be verified experimentally by using different shapes of 'spool dancers' and noting their behavior when spun.

The direction of precession in this case is determined by the following rule:

The external moment causes the positive end of the line $l\sin\theta$ *to precess (rotate about O) toward the positive end of the moment vector,* as indicated in Fig. 29.

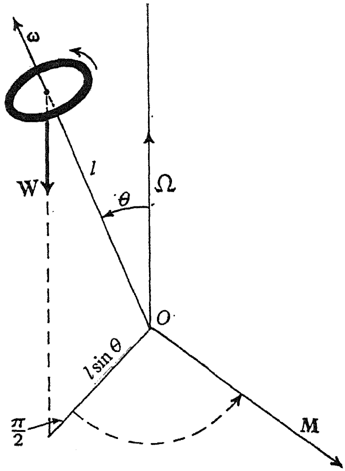


Fig. 29

34. General motion of the top

To investigate the general motion of a top under the action of gravity, we must use equations (32·1). From the third of those equations we get

$$C(\dot{\phi} + \dot{\psi}\cos\theta) = \text{constant} = C\omega, \quad \text{say}. \tag{34·1}$$

Then, since $H_z = C(\dot{\phi} + \dot{\psi}\cos\theta)$, we have

$$H_z = C\omega = \text{constant}.$$

We now write equations (32·1) in the form

$$\left.\begin{aligned} -A\ddot{\psi}\sin\theta - 2A\dot{\theta}\dot{\psi}\cos\theta + C\omega\dot{\theta} &= 0, \\ A\ddot{\theta} + (C\omega - A\dot{\psi}\cos\theta)\,\dot{\psi}\sin\theta &= Wl\sin\theta. \end{aligned}\right\} \tag{34·2}$$

These are the differential equations of motion of the top.

Instead of attempting to integrate these equations directly, we find first integrals of them by other methods.* As the applied force W acts vertically downward, there is no external moment about the vertical axis OZ_1. Hence the angular momentum about OZ_1 is *constant* (Art. 15). The projections on OZ_1 of the angular momenta H_x, H_y and H_z are (Fig. 23) $A\dot{\psi}\sin^2\theta$, 0 and $C\omega\cos\theta$, respectively. Hence we have

$$C\omega\cos\theta + A\dot{\psi}\sin^2\theta = \text{constant} = C_1, \quad \text{say.} \tag{34·3}$$

We find another first integral by utilizing the fact that the total energy of the spinning top is constant. By (17·1) the kinetic energy of a symmetrical gyroscope ($B=A$) is

$$T = \tfrac{1}{2}(A\omega_x^2 + A\omega_y^2 + C\omega_z^2).$$

In our present problem, $\omega_x = -\dot{\psi}\sin\theta$, $\omega_y = \dot{\theta}$ and $\omega_z = \omega$. Hence we have

$$T = \frac{A}{2}(\dot{\theta}^2 + \dot{\psi}^2\sin\theta) + \frac{C\omega^2}{2}.$$

The potential energy at any time with respect to a horizontal plane through O is $Wl\cos\theta$. Hence the total energy is

$$\frac{A}{2}(\dot{\theta}^2 + \dot{\psi}^2\sin\theta) + \frac{C\omega^2}{2} + Wl\cos\theta = \text{constant} = C_2, \quad \text{say.} \tag{34·4}$$

* First integrals of these equations can be found by direct integration as follows:

Multiply the first equation throughout by $-\sin\theta$, thus obtaining

$$(a) \quad A\ddot{\psi}\sin^2\theta + 2A\dot{\theta}\dot{\psi}\sin\theta\cos\theta - C\omega\dot{\theta}\sin\theta = 0.$$

This is an exact differential, and integration gives

$$A\dot{\psi}\sin^2\theta + C\omega\cos\theta = C_1,$$

which is (34·3).

To find a second first integral, multiply equation (a) by $2\dot{\psi}$, multiply the second of equations (34·2) by $2\dot{\theta}$, and add the results thus obtained. Then we have

$$(b) \quad 2A\dot{\theta}\ddot{\theta} + 2A\dot{\psi}\ddot{\psi}\sin^2\theta + 2A\dot{\theta}\dot{\psi}\sin\theta\cos\theta = 2Wl\dot{\theta}\sin\theta.$$

Each member of this equation is now an exact differential, and integration gives

$$A\dot{\theta}^2 + A\dot{\psi}^2\sin\theta = -2Wl\cos\theta + C_2,$$

which is the same as (34·4).

The constants C_1 and C_2 in (34·3) and (34·4) are to be determined from the initial conditions of the problem. If the gyro is released with a spin velocity $\dot{\phi}_0$, the initial conditions are

$$\theta=\theta_0,\quad \dot{\theta}=0 \quad \text{and} \quad \dot{\psi}=0, \quad \text{when } t=0.$$

When these values are substituted into (34·3) and (34·4), we get

$$C_1=C\omega\cos\theta_0,\quad C_2=\frac{C\omega^2}{2}+Wl\cos\theta_0.$$

Hence (34·3) and (34·4) now become

$$A\dot{\psi}\sin^2\theta=C\omega(\cos\theta_0-\cos\theta), \tag{34·5}$$

and

$$A(\dot{\theta}^2+\dot{\psi}^2\sin^2\theta)=2Wl(\cos\theta_0-\cos\theta). \tag{34·6}$$

These two equations determine the subsequent motion of the top.

Equations (34·5) and (34·6) can be reduced to quadratures by the following procedure:

From (34·5) we get

$$\dot{\psi}=\frac{C\omega(\cos\theta_0-\cos\theta)}{A\sin^2\theta}=\frac{C\omega(\cos\theta_0-\cos\theta)}{A(1-\cos^2\theta)}. \tag{34·7}$$

Substituting this value of $\dot{\psi}$ into (34·6), we get

$$A\dot{\theta}^2+A\sin^2\theta\left[\frac{C^2\omega^2(\cos\theta_0-\cos\theta)^2}{A^2\sin^4\theta}\right]=2Wl(\cos\theta_0-\cos\theta),$$

or

$$\dot{\theta}=\frac{1}{A\sin\theta}\sqrt{\{2AWl(\cos\theta_0-\cos\theta)\sin^2\theta-C^2\omega^2(\cos\theta_0-\cos\theta)^2\}}. \tag{34·8}$$

For brevity let us put $\cos\theta=u$. Then

$$-\sin\theta\frac{d\theta}{dt}=\frac{du}{dt} \quad \text{or} \quad \dot{\theta}=-\frac{1}{\sin\theta}\frac{du}{dt}.$$

Hence (34·8) now becomes

$$\frac{du}{dt}=-\frac{1}{A}\sqrt{\{2AWl(u_0-u)(1-u^2)-C^2\omega^2(u_0-u)^2\}},$$

whence

$$\int\frac{du}{\sqrt{\{2AWl(u_0-u)(1-u^2)-C^2\omega^2(u_0-u)^2\}}}=-\frac{t}{A}+C_3. \tag{34·9}$$

This integral on the left is an elliptic integral, and its inversion gives u as a function of t.

Then on writing (34·7) in the form

$$\frac{d\psi}{dt}=\frac{C\omega(u_0-u)}{A(1-u^2)},$$

we get

$$\psi=\frac{C\omega}{A}\int\frac{u_0-u}{1-u^2}\,dt. \tag{34·10}$$

And since u is assumed to be a function of t from (34·9), we can find ψ as a function of t from (34·10).

Finally, from (34·1) we have

$$\frac{d\phi}{dt}+\cos\theta\frac{d\psi}{dt}=\omega.$$

Hence

$$\frac{d\phi}{dt}=\omega-u\frac{d\psi}{dt}=\omega-u\frac{C\omega(u_0-u)}{A(1-u^2)},$$

or

$$\phi=\int\left(\omega-\frac{C\omega}{A}\,\frac{u(u_0-u)}{1-u^2}\right)dt. \tag{34·11}$$

We can therefore find ψ and ϕ as functions of t.

Since elliptic functions are periodic, the above investigation shows that $\cos\theta$, $\dot{\theta}$, $\dot{\psi}$ and $\dot{\phi}$ are all periodic functions of the time and therefore repeat their values at equal intervals of time. As a knowledge of elliptic integrals and elliptic functions is not assumed here, we shall investigate the motion of the spinning axle by other methods.

35. The general precession

To study the precession of the top, we solve (34·5) for $\dot{\psi}$, obtaining

$$\dot{\psi}=\frac{C\omega(\cos\theta_0-\cos\theta)}{A\sin^2\theta}=\frac{C\omega(\cos\theta_0-\cos\theta)}{A(1-\cos^2\theta)}. \tag{35·1}$$

This equation shows that $\dot{\psi}$ and ω have the same sign when $\theta>\theta_0$ and opposite signs when $\theta<\theta_0$, which means that the precession is in the same direction as the spin in the former case and in the opposite direction in the latter case. The equation also shows that $\dot{\psi}=0$ when $\theta=\theta_0$, as it should.

Equation (35·1) shows that when $\theta>\theta_0$, $\dot{\psi}$ has a finite positive value. This means that when the spin axis starts falling, the precession starts. The equation shows further that the precession has a periodic motion because of the periodicity of $\cos\theta$.

On differentiating (35·1) with respect to time, we get

$$\ddot{\psi}=\frac{C\omega\dot{\theta}(1-2\cos\theta_0\cos\theta+\cos^2\theta)}{A\sin^3\theta}. \tag{35·2}$$

This equation (35·2) shows that $\ddot{\psi}=0$ when $\dot{\theta}=0$, but that as soon as $\dot{\theta}$ becomes different from zero $\ddot{\psi}$ has a positive value, because $1-2\cos\theta_0\cos\theta+\cos^2\theta>(1-\cos\theta)^2$. Hence $\dot{\psi}$ begins to increase at once. The train of events is this: The weight W starts to fall, thus increasing θ and giving rise to $\dot{\theta}$; at the same time, $\dot{\psi}$ starts up at an increasing rate.

36. Nutation of the spin axis in the azimuthal plane

To find the motion of the spin axis in the azimuthal plane, we eliminate $\dot{\psi}$ between equations (34·5) and (34·6), thereby obtaining

$$C^2\omega^2(\cos\theta_0-\cos\theta)^2+A^2\dot{\theta}^2\sin^2\theta=2AWl(\cos\theta_0-\cos\theta)\sin^2\theta. \tag{36·1}$$

The maximum and minimum values of θ are found by putting $\dot{\theta}=0$ in (36·1). We thus get

$$(\cos\theta_0-\cos\theta)\,[C^2\omega^2(\cos\theta_0-\cos\theta)-2AWl\sin^2\theta]=0.$$

It follows that $\theta=\theta_0$ and

$$\sin^2\theta=\frac{C^2\omega^2(\cos\theta_0-\cos\theta)}{2AWl}. \tag{36·2}$$

Since $\sin^2\theta$ is always positive, $\cos\theta_0-\cos\theta$ must be positive and $\theta>\theta_0$. This means that θ_0 is the minimum value of θ and that the spin axis is highest in the azimuthal plane for this value.

To find the other values of θ, we put

$$\frac{C^2\omega^2}{2AWl}=2\lambda,$$

and replace $\sin^2\theta$ by $1-\cos^2\theta$. Then (36·2) becomes

$$\cos^2\theta-2\lambda\cos\theta+2\lambda\cos\theta_0-1=0,$$

from which $$\cos\theta=\lambda\pm\sqrt{(1-2\lambda\cos\theta_0+\lambda^2)}.$$

Now since $\cos\theta_0 < 1$, the radicand is greater than $1-2\lambda+\lambda^2$, or $(1-\lambda)^2$. Then for the positive sign before the radical we have

$$\cos\theta > \lambda + (1-\lambda),$$

or

$$\cos\theta > 1,$$

which is impossible. Hence this value of θ must be discarded.

We must therefore take the negative sign before the radical. Then we have

$$\cos\theta_1 = \lambda - \sqrt{(1-2\lambda\cos\theta_0+\lambda^2)}, \tag{36·3}$$

where

$$\lambda = \frac{C^2\omega^2}{4AWl}.$$

Equation (36·3) gives the maximum value of θ and therefore the lowest position of the spin axis in the azimuthal plane. This equation shows that θ_0 and θ_1 are not independent of each other.

From (36·3) we have

$$\cos\theta_1 - \lambda = -\sqrt{(1-2\lambda\cos\theta_0+\lambda^2)}.$$

Squaring this and rearranging the result slightly, we get

$$\cos\theta_1 - \cos\theta_0 = \frac{\cos^2\theta - 1}{2\lambda} = \frac{2AWl(\cos^2\theta-1)}{C^2\omega^2}.$$

The right-hand member of this equation decreases rapidly as the spin velocity increases. Hence $\cos\theta_1 - \cos\theta_0$ likewise decreases at the same rate and consequently $\theta_1 \to \theta_0$. The altitude of the cycloidal loops (Fig. 31) thus decreases rapidly as the spin velocity increases.

If the maximum value of θ_1 is to be $\frac{1}{2}\pi$, thus making the spin axis horizontal, we find from (36·3) that

$$\cos\theta_0 = \frac{1}{2\lambda} = \frac{2AWl}{C^2\omega^2}. \tag{36·4}$$

The angular velocity of the spin axis in the azimuthal plane is found from (36·1) to be

$$\dot\theta = \frac{1}{A\sin\theta}\sqrt{\{2AWl(\cos\theta_0-\cos\theta)\sin^2\theta - C^2\omega^2(\cos\theta_0-\cos\theta)^2\}}. \tag{36·5}$$

Since the radicand is a cubic function in $\cos\theta$, $\cos\theta$ is an elliptic function of t.

37. Path of the spin axis on a unit sphere

Assume that a unit sphere is described about O as center, and that a tangent plane S is drawn through the point P where the axis of spin pierces the sphere (Fig. 30). The velocity of P along the

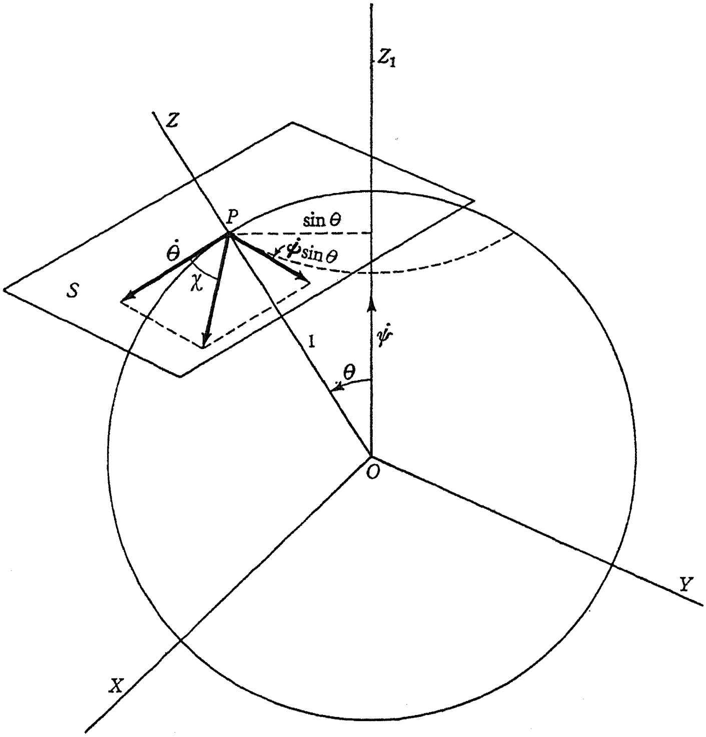


Fig. 30

meridian is therefore $\dot\theta$, and its velocity along the dotted latitude circle is $\dot\psi \sin\theta$.

Let χ denote the angle which the resultant velocity vector makes with the meridian-velocity vector $\dot\theta$. Then $\tan\chi = \dot\psi \sin\theta/\dot\theta$. Now replacing $\dot\psi$ and $\dot\theta$ by their values from (35·1) and (36·5), we obtain

$$\tan\chi = C\omega\sqrt{\frac{\cos\theta_0 - \cos\theta}{2AWl\sin^2\theta - C^2\omega^2(\cos\theta_0 - \cos\theta)}}. \qquad (37{\cdot}1)$$

This equation shows that $\chi = 0$ when $\theta = \theta_0$. The point P is therefore moving along the meridian when $\theta = \theta_0$.

If $\chi = 90°$, so that the motion of P is entirely along the dotted parallel circle, then $\tan\psi = \infty$ and thus the denominator of the fraction in the right-hand member of (37·1) must be zero. Hence we have

$$2AWl\sin^2\theta - C^2\omega^2(\cos\theta_0 - \cos\theta) = 0,$$

or

$$\sin^2\theta = \frac{C^2\omega^2(\cos\theta_0 - \cos\theta)}{2AWl},$$

which is (36·2). Hence P is moving horizontally on the lower circle $\theta = \theta_1$.

By (35·1), $\dot\psi = 0$ when $\theta = \theta_0$; and for $\theta = \theta_1$ the value of $\dot\psi$ is, by (35·1) and (36·2),

$$\dot\psi = \frac{C\omega(\cos\theta_0 - \cos\theta_1)}{A\sin^2\theta} = \frac{C\omega(\cos\theta_0 - \cos\theta_1)}{A}\,\frac{2AWl}{C^2\omega^2(\cos\theta_0 - \cos\theta_1)}$$

$$= \frac{2Wl}{C\omega}.$$

This angular velocity of precession on the lower circle $\theta = \theta_1$ is twice the value found in (33·2) for regular precession.

The preceding discussion has brought out the following facts concerning the motion of the spin axis on the unit sphere:

(1) It reaches its maximum height on the circle $\theta = \theta_0$ and is moving along the meridian when reaching that circle.

(2) It reaches its lowest point on the circle $\theta = \theta_1$ and is moving horizontally at that point.

(3) Equations (35·1) and (36·5) show that the motion is periodic in the azimuthal plane and also along the latitude circles.

These facts enable us to trace the path of the spin axes on the unit sphere as shown in Fig. 31. The path consists of a periodically recurring series of loops which are sometimes called spherical cycloids. A numerical discussion of these loops will be given at the end of the next article.

38. Periods of the cycloidal loops. Pseudo-regular precession

In order to study the motion of the spin axis in more detail, we go back to equation (36·5) and write it in the form

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{\cos\theta_0 - \cos\theta}{A^2\sin^2\theta}\,(2AWl\sin^2\theta - C^2\omega^2(\cos\theta_0 - \cos\theta)).$$

We simplify this by putting $\cos\theta = u$, from which

$$\frac{d\theta}{dt} = -\frac{1}{\sin\theta}\frac{du}{dt}.$$

Then the above equation becomes

$$\left(\frac{du}{dt}\right)^2 = (u_0 - u)\left[\frac{2Wl}{A}(1-u^2) - \frac{C^2\omega^2}{A^2}(u_0 - u)\right].$$

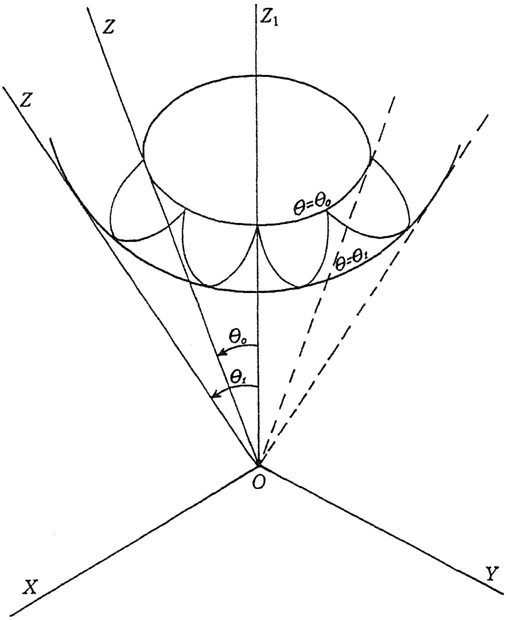


Fig. 31

We simplify it still further by putting

$$a = \frac{2Wl}{A}, \quad b = \frac{C}{A}.$$

Then the equation becomes

$$\left(\frac{du}{dt}\right)^2 = (u_0 - u)\,[a(1-u^2) - b^2\omega^2(u_0 - u)]. \qquad (38\cdot1)$$

Since the right-hand member is a polynomial of the third degree in u, an attempt to integrate (38·1) will lead to an elliptic integral and therefore the inversion of it will give u as an elliptic function

of t; and since elliptic functions are periodic, the rise and fall of the spin axis is periodic. We can avoid the elliptic functions, however, when the spin velocity of the top is high.

Since $(du/dt)^2$ is a positive quantity, the right-hand member of (38·1) must also be positive. The first factor $u_0 - u$ is positive because $\theta_0 < \theta$, and consequently $\cos\theta_0 > \cos\theta$, or $u_0 > u$. The factor in brackets must also be positive, but this fact is not evident at sight. In order to get a better picture of the situation, we construct the graph of equation (38·1).

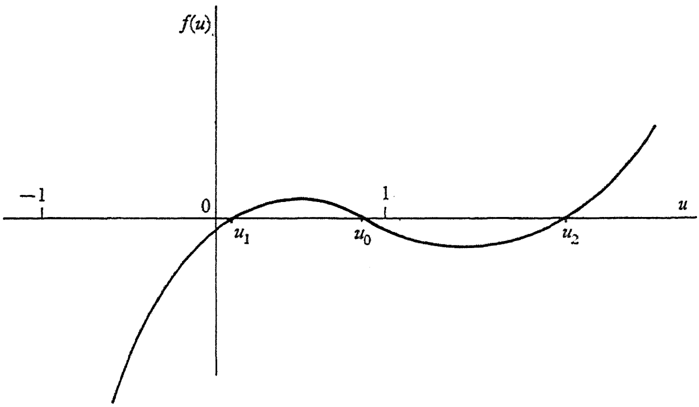


Fig. 32

Let us put $(du/dt)^2 = f(u)$. Then

$$f(u) = a(u_0 - u)(1 - u^2) - b^2\omega^2(u_0 - u)^2. \tag{38·2}$$

Now since au^3 is the dominant term for large values of u, we have

$$f(-\infty) = -\infty, \qquad f(\infty) = \infty,$$
$$f(-1) = -b^2\omega^2(u_0 + 1)^2 = -,$$
$$f(1) = -b^2\omega^2(u - 1)^2 = -.$$

Furthermore, since $\cos\theta$ must lie between -1 and 1, and since $f(u)$ is positive, there must be a region between -1 and 1 where $f(u)$ is positive. These facts enable us to determine the general form of the graph of $f(u)$, which is shown in Fig. 32.

The three roots are thus all real, and the locations of u_0, u_1, u_2, are as shown on the graph. Here u_1 is the value

$$\lambda - \sqrt{(1 - 2\lambda\cos\theta_0 + \lambda^2)},$$

and u_2 is the value $\lambda+\sqrt{(1-2\lambda\cos\theta_0+\lambda^2)}$ found in Art. 36. These two roots can be found by equating to zero the bracketed expression in the right-hand member of (38·1). On putting that expression equal to zero, we have

$$u_0-u=\frac{a(1-u^2)}{b^2\omega^2}. \tag{38·3}$$

This equation shows that when ω is large, u_0-u is very small and is of the order $1/\omega^2$. This fact enables us to find θ and ψ as functions of the time t.

As θ must be very near to θ_0 when ω is large, we may put

$$\theta=\theta_0+\frac{\eta}{\omega^2}, \tag{38·4}$$

where η is an arbitrary variable. Then $\cos\theta=\cos(\theta_0+\eta/\omega^2)$. We now expand $\cos(\theta_0+\eta/\omega^2)$ by the Taylor formula:

$$f(\theta_0+h)=f(\theta_0)+f'(\theta_0)h+\frac{f''(\theta_0)}{2!}h^2+\frac{f'''(\theta_0)}{3!}h^3+\ldots, \tag{38·5}$$

and retain only the first two terms of the expansion. We have

$$\begin{aligned} f(\theta)&=\cos\theta, &&\text{hence}\quad f(\theta_0)=\cos\theta_0;\\ f'(\theta)&=-\sin\theta, &&\text{hence}\quad f'(\theta_0)=-\sin\theta_0;\\ f''(\theta)&=-\cos\theta, &&\text{hence}\quad f''(\theta_0)=-\cos\theta_0. \end{aligned}$$

Now since $h=\eta/\omega^2$, we have by (38·5)

$$\cos\theta=\cos\left(\theta_0+\frac{\eta}{\omega^2}\right)=\cos\theta_0-\frac{\sin\theta_0}{\omega^2}\eta.$$

Hence

$$\left.\begin{aligned} u&=u_0-\frac{\sin\theta_0}{\omega^2}\eta,\\ \frac{du}{dt}&=-\frac{\sin\theta_0}{\omega^2}\frac{d\eta}{dt}. \end{aligned}\right\} \tag{38·6}$$

Substituting into (38·1) these values of u and du/dt, replacing $1-u_0^2$ by $\sin^2\theta_0$, and simplifying slightly, we get

$$\left(\frac{d\eta}{dt}\right)^2=a\omega^2\eta\sin\theta_0+2au_0\eta^2-b^2\omega^2\eta^2.$$

As the term $2au_0\eta^2$ is negligible in comparison with the other two terms (because it does not contain the large factor ω^2), we neglect it and then have

$$\left(\frac{d\eta}{dt}\right)^2 = \omega^2(a\eta\sin\theta_0 - b^2\eta^2),$$

from which

$$\frac{d\eta}{\sqrt{(a\eta\sin\theta_0 - b^2\eta^2)}} = \omega\,dt.$$

Integration of this gives

$$\sin^{-1}\left(\frac{2b^2\eta - a\sin\theta_0}{a\sin\theta_0}\right) = \omega bt + C_1.$$

Now $\theta = \theta_0$ when $t = 0$. Then by (38·4), $\eta = 0$ when $t = 0$. Hence from the above equation we get

$$C_1 = \sin^{-1}(-1) = -\tfrac{1}{2}\pi.$$

Then

$$\sin^{-1}\left(\frac{2b^2\eta - a\sin\theta_0}{a\sin\theta_0}\right) = \omega bt - \tfrac{1}{2}\pi,$$

from which

$$\frac{2b^2\eta - a\sin\theta_0}{a\sin\theta_0} = \sin(\omega bt - \tfrac{1}{2}\pi) = -\cos\omega bt.$$

Solving for η, we get

$$\eta = \frac{a\sin\theta_0}{2b^2}(1 - \cos\omega bt). \tag{38·7}$$

Then by (38·4) we get

$$\theta = \theta_0 + \frac{a\sin\theta_0}{2b^2\omega^2}(1 - \cos\omega bt). \tag{38·8}$$

The period of this motion is

$$T = \frac{2\pi}{\omega b} = \frac{2\pi A}{\omega C}, \tag{38·9}$$

which evidently decreases as the spin velocity increases.

To find $\dot\psi$ as a function of t, we have, from (35·1),

$$\dot\psi = \frac{C\omega(\cos\theta_0 - \cos\theta)}{A(1 - \cos^2\theta)} = \frac{C\omega(u_0 - u)}{A(1 - u^2)}.$$

On replacing u by its value given in (38·6), we have

$$\dot\psi = \frac{C\omega\eta\sin\theta_0}{A(\omega^2\sin^2\theta_0 + 2u_0\eta\sin\theta_0)}.$$

Neglecting the term $2u_0\eta\sin\theta_0$ because of its smallness in comparison with $\omega^2\sin^2\theta_0$, we get

$$\dot{\psi}=\frac{C\eta}{A\omega\sin\theta_0}=\frac{b\eta}{\omega\sin\theta_0},$$

since $C/A=b$. Now replacing η by its value given in (38·7), we have

$$\dot{\psi}=\frac{a}{2b\omega}(1-\cos\omega bt)=\frac{Wl}{C\omega}(1-\cos\omega bt). \tag{38·10}$$

Integration now gives

$$\psi=\frac{Wl}{C\omega}\left(t-\frac{1}{\omega b}\sin\omega bt\right),$$

the constant of integration being zero because $\psi=0$ when $t=0$. Equation (38·10) shows that the angular velocity of precession is the algebraic sum of the steady precession $Wl/C\omega$ (the same as given by (33·2)) and the periodic velocity $(Wl/C\omega)\cos\omega bt$. This periodic term causes the velocity of precession to be slightly unsteady, and for this reason the precession is called *pseudo-regular precession*.

Our results for the general motion of the axis of spin are thus expressed in the two equations

$$\theta=\theta_0+\frac{AWl\sin\theta_0}{C^2\omega^2}(1-\cos\omega bt), \tag{38·8}$$

$$\psi=\frac{Wl}{C\omega}\left(t-\frac{1}{\omega b}\sin\omega bt\right), \tag{38·11}$$

where $b=C/A$. These are the parametric equations of the spherical cycloids in Fig. 31.

Numerical Example. In order to get an idea of the magnitude and frequency of the oscillating motion of the end of the axle of a spinning gyroscope, let us consider a steel disk 6 inches in diameter and 1 inch thick, mounted at the center of a 12-inch spindle of negligible weight. The spindle is held so as to make an angle of 15° with the vertical, its lower end resting in a smooth bearing. If the gyro is given a spin of 7500 r.p.m. and the spindle is free to move in any manner about its point of support, find the rate of precession and the period and amplitude of the upper end of the spindle.

Solution. Before proceeding with the numerical calculation, we

shall derive formulas for A, C and λ in terms of the given quantities. Denoting the mass of the disk by M, its radius by r, and its thickness by h, we have

$$C = \frac{Mr^2}{2},$$

$$A = Ml^2 + M\left(\frac{3r^2+h^2}{12}\right) = \frac{M}{12}(12l^2+3r^2+h^2),$$

$$W = Mg.$$

Substituting these values into the formula,

$$2\lambda = \frac{C^2\omega^2}{2AWl},$$

we get

$$2\lambda = \frac{\dfrac{M^2r^2}{4}\omega^2}{2M\left(\dfrac{12l^2+3r^2+h^2}{12}\right)Mgl} = \frac{3r^4\omega^2}{2(12l^2+3r^2+h^2)gl}.$$

Since, by (36·3), θ_1 depends only on θ_0 and λ, we have the interesting result that θ_1 is independent of the mass of the gyro. This means that the lower circle for a gyro weighing a ton is the same as for one weighing an ounce, if the upper circle is the same in both cases.

We now proceed with the numerical work. The pertinent data in this problem are

$$\omega = 7500 \text{ r.p.m.} = 125 \text{ r.p.s.} = 250\pi \text{ radians per second},$$

$$\theta_0 = 15°,$$

$$l = \tfrac{1}{2} \text{ foot}, \quad r = \tfrac{1}{4} \text{ foot}, \quad h = \tfrac{1}{12} \text{ foot}, \quad g = 32{\cdot}16 \text{ ft./sec.}^2$$

Substituting these data into the expression for 2λ above, we have

$$2\lambda = \frac{3(\frac{1}{4})^4(250\pi)^2}{2[12(\frac{1}{2})^2+3(\frac{1}{4})^2+(\frac{1}{12})^2](32.16)(\frac{1}{2})} = 70.36387.$$

$$\therefore \quad \lambda = 35.18194.$$

Now substituting into (36·3) the values of θ_0 and λ, we have

$$\cos\theta_1 = 35.18194 - 34.21699 = 0.96495.$$

$$\therefore \quad \theta_1 = 15°\, 12'\, 52''.$$

The amplitude of nutation in the azimuthal plane is thus only

$$\theta_1 - \theta_0 = 12' \, 52'' = 0.00374 \text{ radian.}$$

The arc distance between the parallel circles θ_0 and θ_1 on the unit sphere is thus

$$s = r(\theta_1 - \theta_0) = 1 \times 0.00374 = 0.00374 \text{ foot}$$

$$= 0.0449 \text{ inch} = \tfrac{1}{22} \text{ inch.}$$

The top of the spindle thus oscillates in a zone $\frac{1}{22}$ inch wide on the unit sphere and deviates only about $\frac{1}{50}$ inch from its mean precessional path.

From the data of the problem we have

$$\frac{A}{C} = \frac{\dfrac{M}{12}(12l^2 + 3r^2 + h^2)}{\dfrac{Mr^2}{2}} = \frac{1}{6}\left(\frac{12l^2 + 3r^2 + h^2}{r^2}\right)$$

$$= \frac{1}{6}\left(\frac{12(\frac{1}{2})^2 + 3(\frac{1}{4})^2 + (\frac{1}{12})^2}{(\frac{1}{4})^2}\right) = \frac{230}{27}.$$

Hence by (38·9) the period of the oscillations is

$$T = \frac{2\pi}{250\pi} \times \frac{230}{27} = 0.068 \text{ second,}$$

which means 14.7 vibrations per second, or about three times as fast as the ticking of a watch.

By (38·10) the steady or constant part of the angular velocity of precession is

$$\dot{\psi} = \frac{a}{2b\omega} = \frac{Wl}{C\omega} = \frac{Mgl}{(Mr^2/2)\,\omega} = \frac{2gl}{r^2\omega}$$

$$= \frac{2(32.16)\,(\frac{1}{2})}{(\frac{1}{4})^2\,(250\pi)} = 0.655 \text{ radian per second.}$$

This means one revolution in 9.6 seconds, or 6.26 r.p.m.

The number of cycloidal loops on the unit sphere is thus

$$\frac{9.6}{0.068} = 141.$$

The circumference of the upper circle θ_0 is

$$2\pi(12 \sin 15°) = 19.5 \text{ inches.}$$

Hence the width (base) of each loop is

$$\frac{19.5}{141} = 0.138 \text{ inch.}$$

Since the base of the ordinary plane cycloid is π times its maximum height, the base of a cycloidal loop 0.0449 inch high would be 0.0449π, or 0.141 inch. The spherical cycloids of the present example are therefore slightly more slender than plane cycloids.

The results of this example show that the axle of a spinning top precesses with a rapid tremor of small amplitude, the tremor being hardly noticeable to the naked eye.

39. The sleeping top

When a top is spun rapidly with its axis nearly vertical and then released, it soon settles down to a steady spin with its axis seemingly vertical. The top in this apparently motionless state is said to be 'sleeping'. We now proceed to determine the necessary condition for the top to sleep.

On substituting into the second of equations (34·2) the value of $\dot{\psi}$ given by (35·1) and then replacing $\sin^2\theta + \cos^2\theta$ by 1, we get

$$A\ddot{\theta} + \frac{C^2\omega^2(1-\cos\theta_0\cos\theta)(\cos\theta_0 - \cos\theta)}{A\sin^3\theta} = Wl\sin\theta. \quad (39{\cdot}1)$$

Now since the spin axis of the top is practically vertical when the top is sleeping, we put $\theta_0 = 0$. Then (39·1) becomes

$$A\ddot{\theta} + \frac{C^2\omega^2(1-\cos\theta)^2}{A\sin^3\theta} = Wl\sin\theta. \quad (39{\cdot}2)$$

Here θ is a very small angle, and this fact enables us to simplify the equation still further. From the series expansion

$$\cos\theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \ldots,$$

we get
$$(1-\cos\theta)^2 = \frac{\theta^4}{4}.$$

Then after replacing $\sin\theta$ by θ, equation (39·2) becomes

$$A\ddot{\theta}+\frac{C^2\omega^2\theta^4}{4A\theta^3}=Wl\theta,$$

or
$$\ddot{\theta}+\frac{C^2\omega^2-4WAl}{4A^2}\theta=0. \tag{39·3}$$

This is a harmonic motion equation, the solution of which is

$$\theta=C_1\cos kt+C_2\sin kt,$$

where
$$k=\frac{\sqrt{(C^2\omega^2-4WAl)}}{2A}.$$

Then we get
$$\dot{\theta}=-C_1k\sin kt+C_2k\cos kt. \tag{39·4}$$

To find the value of $\dot{\psi}$ for a sleeping top we put $\theta_0=0$ in equation (35·1). Then we have

$$\dot{\psi}=\frac{C\omega(1-\cos\theta)}{A\sin^2\theta}=\frac{C\omega(1-\cos\theta)}{A(1-\cos\theta)(1+\cos\theta)},$$

or
$$\dot{\psi}=\frac{C\omega}{A(1+\cos\theta)}. \tag{39·5}$$

Equations (39·4) and (39·5) show that when a top is sleeping the free end of the spin axis moves with simple harmonic motion in the azimuthal plane and that it precesses with a periodic motion about the vertical. It is thus a stable motion about the vertical position. If we let $\theta\to 0$ in (39·5), we see that $\dot{\psi}\to C\omega/2A$, a constant. The irregularity in the precession thus fades out as the spin axis becomes vertical.

The period of the motion of the sleeping top is

$$P=\frac{2\pi}{k}=\frac{4\pi A}{\sqrt{(C^2\omega^2-4WAl)}}.$$

If this period is to be real, that is, if the spin axis is actually to oscillate and the motion is to be stable, we must have

$$C^2\omega^2>4WAl,$$

or
$$\omega^2>\frac{4WAl}{C^2}. \tag{39·6}$$

Equation (39·6) thus gives the velocity of spin necessary for a stable sleeping top. A smaller spin velocity will allow the top to wobble

and eventually fall. This equation shows that if C is small in comparison with A, the spin velocity must be very high in order for the top to sleep. This fact explains why it is almost impossible to spin a pencil on its point.

For the case of a sleeping top the graph of $f(u)$ has the appearance shown in Fig. 33, the roots u_0 and u_1 each being equal to 1.

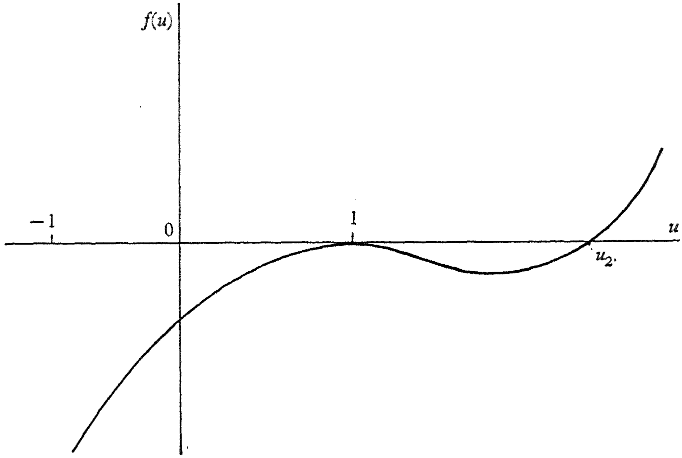


Fig. 33

Historical note

The early history of the gyroscope is rather obscure. Probably the earliest gyroscope of the type now in use was constructed by Bohnenberger, of Germany, about 1810. An improved type was constructed by Walter Rogers Johnson, of Philadelphia, Pa., in 1832, and used by him to illustrate the dynamics of rotating bodies. Johnson called his device a 'rotascope'.

In 1852 Léon Foucault, of Paris, constructed a very refined gyroscope to show the rotation of the earth. In order to bring the rotor to very high speed and keep it spinning for several minutes, Foucault constructed a gear train machine on which the gyro was placed for speeding up. When the speed was sufficiently high, the spinning gyro was lifted from the machine and placed in its gimbals. Because of the fact that the refined device exhibited the rotation of the earth before his eyes, Foucault named it the 'gyroscope', from the Greek words *gyros* (γύρος), circle or ring, and *skopein* (σκοπεῖν), to view.

In 1878 George M. Hopkins, American inventor, applied the electric motor to the gyroscope and thereby kept it spinning at any desired speed as long as desired. Gyroscopes were not put to practical and industrial use until the first decade of the twentieth century, at which time the gyroscopic compass, the ship stabilizer, and the monorail car were all invented.

The mathematical foundations of gyroscopic theory were laid by Leonhard Euler in 1765, in his *Theoria motus corporum solidorum sue rigidorum*.

PART II

APPLICATIONS OF THE GYROSCOPE

CHAPTER V

Gyroscopic Action in Vehicles and Rotating Bodies

Because of the stubborn tendency of a rapidly spinning gyro to maintain its plane of rotation, gyroscopes are used as direction indicators and as stabilizers. The forced precession of the gyro axle also produces interesting reactionary effects in rotating machinery and in the astronomical world.

40. Gyroscopic effects in car wheels rounding a curve

When a vehicle such as a train, automobile or racing car is moving at high speed along a straight road, the gyroscopic effect of the wheels is to keep the vehicle moving straight ahead. But when an external moment is applied to force the vehicle to change its direction, gyroscopic reaction immediately comes into play. To determine this gyroscopic reaction in the case of vehicles, let us consider a pair of wheels connected by a straight axle. This wheel-axle ensemble may be considered a gyroscope. When the vehicle is forced to round a curve, such rounding is a forced precession of the wheel-axle ensemble around a vertical axis through the center of curvature of the curved track. For simplicity we shall assume that the roadbed is not banked and, in the case of a railroad, that the outer rail on the curve is not elevated higher than the inner rail.

Let Fig. 34 represent an axle and its two wheels rounding a circular curve of radius R with speed v. Let O be the center of curvature of the curve, and let r denote the radius of the wheels. If ω denotes the angular velocity of the wheels and Ω denotes the angular velocity of precession about the vertical axis OV, then

$$v = R\Omega, \tag{40·1}$$

and

$$v = r\omega. \tag{40·2}$$

By equation (27·2), with $\theta = 90°$, the gyroscopic reaction moment due to the forced precession is

$$K = C\omega\Omega, \tag{40·3}$$

where C denotes the moment of inertia of both wheels about their common axle and K is the reaction couple (moment) acting in the

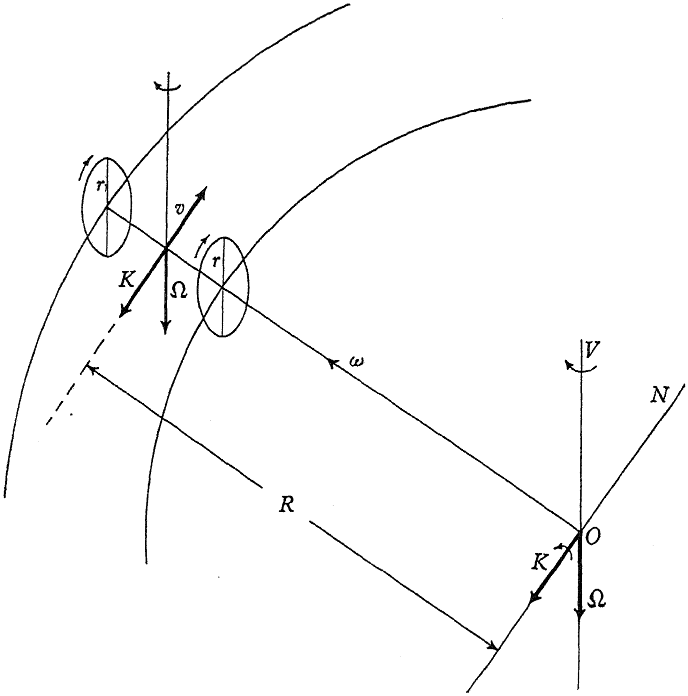


Fig. 34

vertical plane through OV and the axle. Substituting into (40·3) the values of Ω and ω from (40·1) and (40·2), we get

$$K = \frac{Cv^2}{Rr}. \tag{40·4}$$

The gyroscopic reaction thus varies as the square of the velocity of the vehicle along the curve.

Since gyroscopic phenomena depend only on rotation and not at all on translation, and since the angular velocity of precession of the ensemble about a vertical axis through the mid-point of the

axle is exactly the same as its velocity of precession about OV, we may consider all the gyroscopic phenomena as taking place with reference to the mid-point of the axle as origin. Then by the right-handed-screw rule the precession vector must be drawn vertically downward; and since the gyroscopic reaction moment tends to turn the spin vector into coincidence with the precession vector (Art. 28), we see at once that the reaction vector **K** must be drawn backward, as indicated in Fig. 34. This shows that the effect of the gyroscopic reaction is to decrease the pressure on the inner rail or wheel and increase it on the outer rail or wheel. The gyroscopic moment is thus in the same direction as the moment due to centrifugal force and aids the latter in its tendency to overturn the vehicle.

In order to compare the gyroscopic and centrifugal moments, let us consider a vehicle having n pairs of wheels. Let

M = mass of entire vehicle, including wheels.

m = mass of each wheel.

h = height of c.g. of vehicle above roadbed.

r = radius of each wheel (= distance from center to rail tread in the case of locomotive and train wheels).

c = a constant such that cr = radius of gyration. For most wheels, $1/\sqrt{2} \leqslant c < 1$.

Then for a single wheel the moment of inertia about the axle is mcr^2, and for n pairs of wheels the moment of inertia is

$$C = 2mncr^2.$$

By (40·4) the gyroscopic reaction moment for n pairs of wheels is therefore

$$K = \frac{2mncr^2v^2}{Rr}.$$

Since the moment of the centrifugal force about the outer rail is

$$N = \frac{Mv^2}{R}h,$$

we have

$$\frac{K}{N} = \frac{2mncr}{Mh} = c\left(\frac{2mn}{M}\right)\left(\frac{r}{h}\right). \tag{40·5}$$

Now since the mass of the wheels is less than the mass of the entire vehicle and h is greater than r, we have $2mn/M < 1$, $r/h < 1$, $c < 1$. Hence

$$\frac{K}{N} < 1,$$

or

$$K < N.$$

Note that the ratio K/N is independent of the speed of the vehicle and of the radius of the curve. In all ordinary vehicles the gyroscopic moment is so much less than the centrifugal moment that the effect of the former need not be considered. In automobiles, for example, the gyroscopic moment constitutes less than 5 % of the total overturning moment when rounding a curve. Calculations from known data for one electric locomotive showed that the gyroscopic moment was only 6 % of the centrifugal moment, or about $5\frac{2}{3}$ % of the total overturning moment.

Although the gyroscopic and centrifugal moments act in the same direction, the forces of those moments do not have the same direction. The forces due to gyroscopic moments act vertically downward on the outer rail or wheel and vertically upward in the case of the inner rail or wheel, whereas the centrifugal force acts horizontally outward on both wheels.

In order to make a quantitative comparison of the gyroscopic and centrifugal moments on a vehicle, let us consider the extreme case of a racing car. Assume that the wheels are 3 feet in diameter and weigh 100 pounds each, that the combined weight of car and driver is 1800 pounds, that the center of gravity of car and driver is 20 inches above the ground, and that $c = 0{\cdot}9$. Then

$$r = \tfrac{3}{2} \text{ feet}, \quad h = \tfrac{20}{12} = \tfrac{5}{3} \text{ feet}, \quad M = 1800/g, \quad m = 100/g, \quad n = 2.$$

Substituting these data into (40·5), we get

$$\frac{K}{N} = \frac{9}{10}\left(\frac{400/g}{1800/g}\right)\frac{\frac{3}{2}}{\frac{5}{3}} = 0.18,$$

or

$$K = 0.18N.$$

The gyroscopic moment is thus less than one-fifth of the centrifugal moment.

Another gyroscopic effect in automobiles is that due to the flywheel. Since the axis of rotation of the flywheel is along the longitudinal axis of the chassis and since the flywheel rotates to the

left as viewed from behind, the spin vector must be drawn backward along the axis of spin. When the car makes a turn to the right or to the left, the axle of the flywheel is forced to precess in the same direction about a vertical axis. This precession brings into play a gyroscopic reaction moment about a transverse horizontal axis. The effect of this reaction moment is to increase the downward pressure on one end of the flywheel axle and increase the upward pressure on the other end, thus tending to bend the axle in a vertical plane. The gyroscopic moment in this case is therefore a bending moment.

The magnitude of this gyroscopic moment is given by the usual formula

$$K = C\Omega\omega,$$

where ω is the angular velocity of the flywheel, Ω is the angular velocity of precession, and C is the moment of inertia of the flywheel about its axis of spin. The gyroscopic reaction moment due to the precession of the spin axle of the flywheel in automobiles is usually of minor importance because of the relatively small mass of the flywheel.

41. Derivation of the differential equations of motion of a gyroscope by application of the principle of gyroscopic reaction

In the applications of the gyroscope the differential equations of motion can be derived in either of two ways: (1) by determining the angular velocities ω_x, ω_y, ω_z about the axes of the moving trihedron and then using equations (16·4), or (2) by determining the G.R.M.'s and then substituting them in the fundamental rotational equation (20·2). One method is preferable in some problems and the other in others. The G.R.M. method is the shorter, but is less foolproof. The reader is advised to use the first method until he feels safe in using the second.

As an illustration of the G.R.M. method, let Fig. 35 represent a gyro spinning about a horizontal axis and let the axle be acted on by a horizontal couple **F**, $-$**F**. Let us consider the moments about the axes OZ and OX.

(a) For OZ we have first the moment **M** due to the external couple **F**, $-$**F**. This couple tends to increase the angle ϕ and is therefore positive and must have the same sign as $d^2\phi/dt^2$. The couple also causes the axle of spin to precess about the horizontal axis OX as indicated in the figure. This forced precession about OX

induces a gyroscopic reaction moment $C\omega\, d\theta/dt$ about OZ, this moment being negative according to the rule laid down in Art. 28. Hence the reaction vector **K** must be drawn downward. The total moment about OZ is therefore $M - C\omega\, d\theta/dt$. Then from the funda-

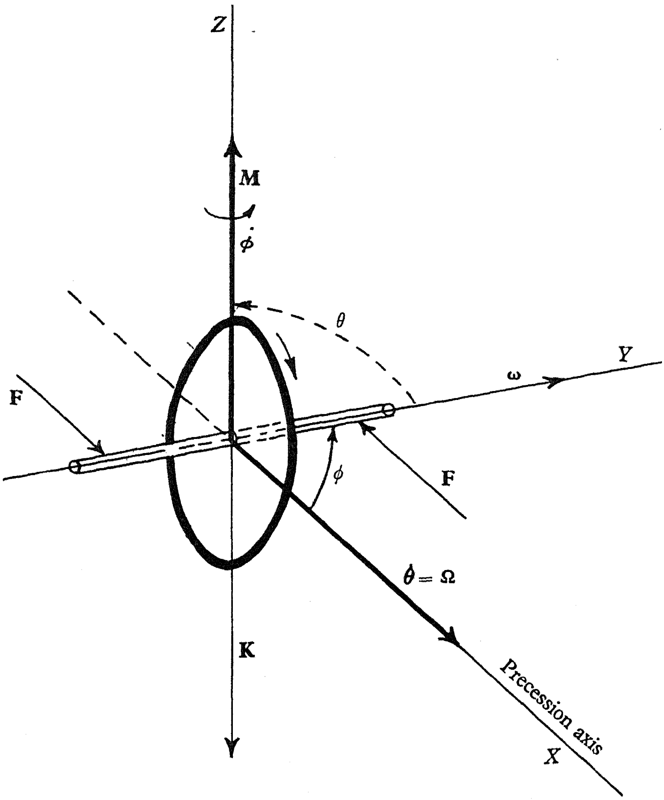


Fig. 35

mental moment equation (20·2) we have for the motion about OZ the differential equation

$$A\frac{d^2\phi}{dt^2} = M - C\omega\frac{d\theta}{dt},$$

or

$$A\frac{d^2\phi}{dt^2} + C\omega\frac{d\theta}{dt} = M,$$

where A denotes the moment of inertia of the gyro about OZ.

(*b*) As to the moments about OX, we see that there is no external moment. The first effect of the external couple **F**, $-$**F** is to cause an angular velocity $d\phi/dt$ about OZ. This forced rotation (a precession) will immediately induce a gyroscopic reaction moment of magnitude $C\omega\, d\phi/dt$ about OX, and by the rule of Art. 28 this moment is positive. Hence from the fundamental equation for rotation about an axis we have

$$A\frac{d^2\theta}{dt^2}=C\omega\frac{d\phi}{dt},$$

or

$$A\frac{d^2\theta}{dt^2}-C\omega\frac{d\phi}{dt}=0,$$

for the motion about OX.

The differential equations for the motion of the gyro under the action of external couple **F**, $-$**F** are therefore

$$A\frac{d^2\theta}{dt^2}-C\omega\frac{d\phi}{dt}=0,$$

$$A\frac{d^2\phi}{dt^2}+C\omega\frac{d\theta}{dt}=M,$$

where **M** is the external moment tending to deflect the axis of spin.

If we derive the differential equations of motion for this example by using equations (16·4), we have

$$\begin{aligned}
&\omega_x=\dot\theta, &&H_x=A\dot\theta, &&dH_x/dt=A\ddot\theta,\\
&\omega_y=0, &&H_y=C(0+\omega)=C\omega, &&dH_y/dt=0,\\
&\omega_z=\dot\phi, &&H_z=A\dot\phi, &&dH_z/dt=A\ddot\phi.
\end{aligned}$$

On substituting these into (16·4), we have

$$A\ddot\theta-C\omega\dot\phi=M_x=0,$$

$$0+A\dot\theta\dot\phi-A\dot\phi\dot\theta=M_y,$$

$$A\ddot\phi+C\omega\dot\theta=M_z=M,$$

or

$$A\frac{d^2\theta}{dt^2}-C\omega\frac{d\phi}{dt}=0,$$

$$A\frac{d^2\phi}{dt^2}+C\omega\frac{d\theta}{dt}=M,$$

as before.

42. Gyroscopic effects of dynamos, motors and turbines installed on ships

The rolling, pitching and yawing of a ship on a rough sea can cause gyroscopic action of motors and turbines in three mutually perpendicular directions. Rolling is rotation about the longitudinal

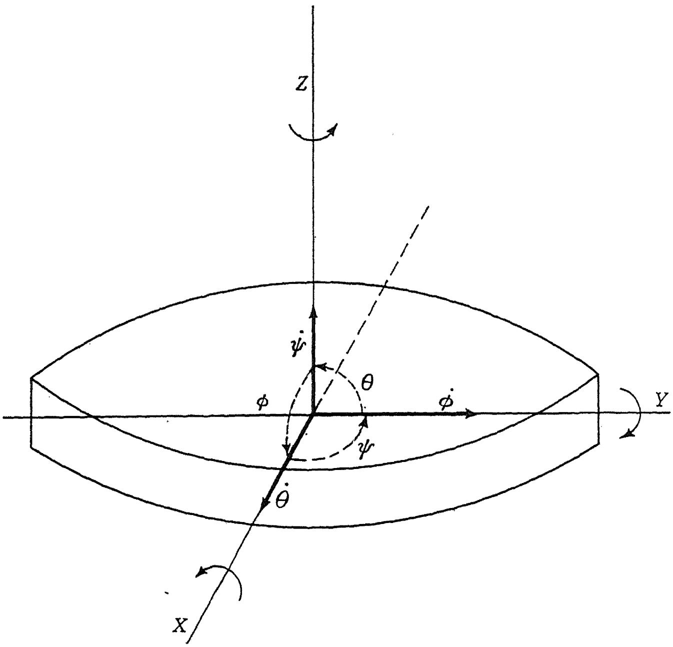


Fig. 36

axis of the ship, pitching is rotation about a transverse axis, and yawing is rotation about a vertical axis, all these axes intersecting at the center of gravity of the ship as origin. Let us consider the gyroscopic effects of these actions on a motor or a turbine when its axis of spin is (*a*) parallel to the longitudinal axis of the ship, (*b*) parallel to the deck and perpendicular to the longitudinal axis, and (*c*) perpendicular to the deck. Bearing in mind that each of the above motions of the ship will cause a forced precession of the axis of spin, applying the G.R.M. method of the preceding article or the formulas of Art. 27 and rule of Art. 28, and referring to Fig. 36, we obtain the following results:

(*a*) *Axis of spin parallel to longitudinal axis of ship:*

Rolling: no gyroscopic action.

Yawing: $K_x = C\omega \dfrac{d\psi}{dt}$.

Pitching: $K_z = -C\omega \dfrac{d\theta}{dt}$.

(*b*) *Axis of spin transverse to axis of ship:*

Rolling: $K_z = C\omega \dfrac{d\phi}{dt}$.

Yawing: $K_y = -C\omega \dfrac{d\psi}{dt}$.

Pitching: no gyroscopic action.

(*c*) *Axis of spin perpendicular to deck:*

Rolling: $K_x = -C\omega \dfrac{d\phi}{dt}$.

Yawing: no gyroscopic reaction.

Pitching: $K_y = C\omega \dfrac{d\theta}{dt}$.

In the above formulas C denotes the moment of inertia of the gyro about its axis of spin.

Since rolling is the most common form of motion of a ship, it is very important that motors and turbines be installed with their spin axes parallel to the longitudinal axis of the ship, for in that position rolling does not cause any gyroscopic action.

To get a quantitative estimate of the gyroscopic moment due to rolling and pitching, let α denote the angle which the ship mast makes with the vertical (the plumb line) at any time t, and let Φ be the maximum numerical value of α. Then since rolling and pitching are periodic phenomena, we may write

$$\alpha = \Phi \sin \beta, \tag{42·1}$$

where β is an auxiliary variable such that $\beta = \frac{1}{2}\pi$ when $\alpha = \Phi$. Then for the angular velocity of rolling or pitching, we have

$$\frac{d\alpha}{dt} = \Phi \cos \beta \frac{d\beta}{dt}. \tag{42·2}$$

In terms of β the period, T, of rolling is given by the well-known formula

$$T=\frac{2\pi}{d\beta/dt}.$$

Hence
$$\frac{d\beta}{dt}=\frac{2\pi}{T}.$$

Substituting into (42·2) this value of $d\beta/dt$, we get

$$\frac{d\alpha}{dt}=\frac{2\pi\Phi}{T}\cos\beta. \tag{42·3}$$

This angular velocity has its maximum value when $\beta=0$. Hence

$$\left(\frac{d\alpha}{dt}\right)_{\max}=\frac{2\pi\Phi}{T}. \tag{42·4}$$

Now since the rolling and pitching of the ship forces the axles of dynamos and turbines to precess with the same angular velocity, we have from (42·4)

$$\Omega=\frac{2\pi\Phi}{T}, \tag{42·5}$$

where Ω denotes the angular velocity of precession.

If C denotes the moment of inertia of the turbine rotor about its axis of spin and ω is the spin velocity, we have

$$K=C\Omega\omega=\frac{2\pi\Phi C\omega}{T} \tag{42·6}$$

for the gyroscopic reaction moment due to the rolling or pitching of the ship.

Numerical Example. Let us assume that a steam turbine weighs 3 tons and is rotating at 3000 r.p.m. or 100π radians per second. Then

(a) If the turbine were installed with its axis *at right angles* to the longitudinal axis of the ship, the gyroscopic reactions would be due to the rolling of the ship. If we assume an angle of roll of 20° to either side from the vertical and that the period of rolling is 8 seconds, then

$$\Phi=20^\circ=\frac{\pi}{9}\text{ radians}$$

and
$$T=8\text{ seconds}.$$

If the average radius of the turbine rotor is 1 foot and the moment of inertia is assumed to be $\frac{3}{4}Mr^2$, then by (42·6)

$$K = \frac{2\pi\left(\frac{\pi}{9}\right)\left(\frac{3}{4}\right)\left(\frac{6000}{32.2}\right)(100\pi)}{8} = 12{,}036 \text{ lb.ft.}$$

This reaction moment reverses its direction every 4 seconds. It acts in a plane parallel to the deck of the ship and tends to skew the axle of the turbine back and forth ever 4 seconds, thus exerting a pulsating pressure against the sides of the bearings. Since the rotor is rigidly keyed to its axle, the axle is subjected to an alternating bending moment which changes its direction (with respect to a diametral plane of the axle) 3000 times per minute.

(*b*) If the turbine be installed with its spin axes *parallel* to the longitudinal axis of the ship, as it should be, then the only gyroscopic action of any importance that will occur is that due to the pitching of the ship. The angle of pitching is less than the angle of rolling and will be taken as 10° in the present example. The period of pitching will be taken as 10 seconds. Then

$$\Phi = 10^\circ = \frac{\pi}{18} \text{ radians,}$$

$$T = 10 \text{ seconds.}$$

Hence by (42·6) we have

$$K = \frac{2\pi\left(\frac{\pi}{18}\right)\left(\frac{3}{4}\right)\left(\frac{6000}{32.2}\right)(100\pi)}{10} = 4815 \text{ lb.ft.}$$

The gyroscopic moment due to pitching is thus very much smaller than that due to rolling, because of the smaller angle in the case of pitching.

(*c*) If the ship turns completely around (through 180°) in 1 minute, the angular velocity of precession is $\Omega = \frac{1}{60}\pi$ radian per second. Hence the gyroscopic reaction moment in this case is

$$K = C\Omega\omega = \frac{3}{4}\left(\frac{6000}{32.2}\right)\left(\frac{\pi}{60}\right)(100\pi) = 229{\cdot}8 \text{ lb.ft.}$$

This moment acts in a plane perpendicular to the deck. If the turbine is installed with its axis parallel to the longitudinal axis of

the ship, the above reaction moment will act in a perpendicular plane which contains the longitudinal axis and will raise one end of the ship and depress the other. For example, if the turbine is rotating to the right as viewed from the rear end of the axle, the gyroscopic moment will raise the stern if the ship turns around to the right and raise the bow if it turns around to the left. Similar gyroscopic effects can be observed when making a quick turn with a motor boat.

43. Rolling hoops and disks

A rolling hoop or disk (coin, for example) may be looked upon as a gyroscope whose center of gravity has an irregular motion of translation and whose rotor rolls on a rough surface without slipping. Such a motion is more complex than that of the usual type of gyroscope, and its mathematical treatment is considerably more complicated. Inasmuch as the mathematical treatment of the most general case of hoop motion would take us beyond the scope of this book, we do not enter into it. Instead, we confine our treatment to some of the simpler and more familiar aspects of the motion.*

Everyday observation shows that if a hoop is started rolling on the ground in a vertical plane, it will steer itself for a time as it encounters small obstructions in its path. Thus if it passes over a small obstruction which tilts it from a vertical plane, the moment of its weight immediately causes the hoop to precess about a vertical axis through the point of contact (about a vertical diameter, practically) and thus steers it in the direction in which it started to fall. This change in direction sets up a centrifugal force which tends to bring the hoop back to a vertical position and allow it to roll on in the new direction.

To study the behavior of a rolling hoop, let us assume that it is started rolling with constant angular velocity ω. Fig. 37 represents it in a slightly inclined position after encountering a small obstacle in its path. This slight tilt causes it to travel temporarily in a curved path. Let MN denote the tangent to the path at the point of contact of the hoop, and let $d\phi$ denote the horizontal angle through

* The interested reader will find a good treatment of the general case in Appell's *Traité de Mécanique Rationnelle*, tome 2 (third edition, 1911), pp. 248–54.

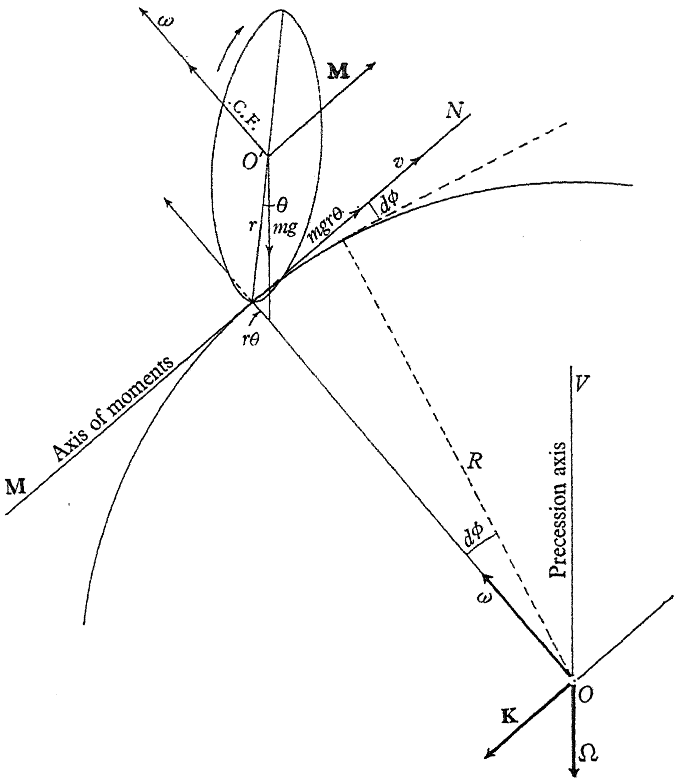


Fig. 37

which the plane of the hoop turns in time dt. Then the angular velocity of the precession of the hoop about a vertical axis is $d\phi/dt$. Let

ω = angular velocity of hoop about its geometric axis,

v = linear velocity of point of contact
= linear velocity of c.g. of hoop,

r = radius of hoop,

R = radius of curvature of path,

θ = angle of inclination of plane of hoop with the vertical,

m = mass of hoop,

C = moment of inertia of hoop about its geometric axis,

A = moment of inertia of hoop about a diameter,

I_{MN} = moment of inertia of hoop about MN.

Then $$v = r\omega \tag{43·1}$$

and $$v = R\frac{d\phi}{dt}. \tag{43·2}$$

Since ω is assumed to be constant, v must also be constant. Hence

$$R\frac{d\phi}{dt} = \text{a constant}. \tag{43·3}$$

Now let us consider the moments acting about the line MN when the plane of the hoop is inclined at a small angle θ. The moment of the weight is $mgr \sin\theta$, which is practically $mgr\theta$, since θ is assumed to be small. This moment tends to increase θ.

The moment of the centrifugal force is

$$r\frac{mv^2}{R} = rmv\frac{v}{R} = rmv\frac{d\phi}{dt} \quad \text{(by (43·2))} = rmr\omega\frac{d\phi}{dt} = mr^2\omega\frac{d\phi}{dt},$$

and this moment tends to decrease θ.

The precession of the hoop about a vertical axis induces a gyroscopic reaction moment of magnitude $C\omega(d\phi/dt)$. This moment also tends to decrease θ. Hence from the fundamental moment equation (20·2) we have

$$I_{MN}\frac{d^2\theta}{dt^2} = mgr\theta - mr^2\omega\frac{d\phi}{dt} - C\omega\frac{d\phi}{dt}.$$

We next consider the moments about a vertical diameter through the point of contact. The only moment about this line is the gyroscopic reaction moment tending to increase ϕ to keep the hoop from falling. This moment is $C\omega(d\theta/dt)$, and it has the same sign as $d^2\phi/dt^2$. Hence by the fundamental moment equation we have

$$A\frac{d^2\phi}{dt^2} = C\omega\frac{d\theta}{dt}.$$

The differential equations for the dynamical equilibrium of the hoop are therefore

$$\left.\begin{aligned} I_{MN}\frac{d^2\theta}{dt^2} + mr^2\omega\frac{d\phi}{dt} + C\omega\frac{d\phi}{dt} &= mgr\theta, \\ A\frac{d^2\phi}{dt^2} - C\omega\frac{d\theta}{dt} &= 0. \end{aligned}\right\} \tag{43·4}$$

These equations will be solved with the initial conditions $\theta=0$ and $\dot{\theta}=\dot{\theta}_0$ when $t=0$. In this case the hoop starts out in a vertical plane. Since ω is assumed to be constant, we can integrate the second equation at once, thereby obtaining

$$A\frac{d\phi}{dt}-C\omega\theta=\text{constant}=C_1, \quad \text{say.}$$

If $d\phi/dt=0$ when $\theta=0$, we get $C_1=0$. Hence we have

$$A\frac{d\phi}{dt}=C\omega\theta. \tag{43·5}$$

This equation states that the rate of change of direction in its path is proportional to the inclination of the hoop, and this, in view of equation (43·3), means that the more the hoop or coin is inclined, the greater is the curvature of its path. The behavior of rolling coins agrees with this statement.

On substituting in the first of equations (43·4) the value of $d\phi/dt$ from (43·5), we get

$$I_{MN}\frac{d^2\theta}{dt^2}+\left(\frac{Cmr^2\omega^2}{A}+\frac{C^2\omega^2}{A}-mgr\right)\theta=0. \tag{43·6}$$

For a *hoop* we have

$$C=mr^2, \quad A=\frac{mr^2}{2}, \quad I_{MN}=mr^2+\frac{mr^2}{2}=\frac{3mr^2}{2}.$$

Substituting these values into (43·6), we get

$$\frac{d^2\theta}{dt^2}+\frac{2}{3}\left(4\omega^2-\frac{g}{r}\right)\theta=0. \tag{43·7}$$

The auxiliary equation of this differential equation is

$$\alpha^2+\frac{2}{3}\left(4\omega^2-\frac{g}{r}\right)=0,$$

or

$$\alpha^2=-\frac{2}{3}\left(4\omega^2-\frac{g}{r}\right). \tag{43·8}$$

If the motion of the hoop is to be stable in a vertical plane, so that it will oscillate about a vertical position as it rolls along, the values of α must be imaginary. Hence we must have

$$4\omega^2>\frac{g}{r},$$

or

$$\omega^2 > \frac{g}{4r}. \tag{43·9}$$

If the spin velocity becomes less than $\frac{1}{2}\sqrt{(g/r)}$, the hoop will begin to wobble and will soon fall over.

From (43·8) we have

$$\alpha = \pm i\sqrt{\left\{\frac{2}{3}\left(4\omega^2 - \frac{g}{r}\right)\right\}} = \pm ik, \quad \text{say.}$$

Consequently, the general solution of (43·7) is

$$\theta = C_2 \sin kt + C_3 \cos kt.$$

Then

$$\dot{\theta} = C_2 k \cos kt - C_3 k \sin kt.$$

Now substituting the initial conditions into these equations, we find

$$C_3 = 0, \quad C_2 = \frac{\dot{\theta}_0}{k}.$$

Hence the values of θ and $\dot{\theta}$ are

$$\theta = \frac{\dot{\theta}_0}{k} \sin kt, \tag{43·10}$$

$$\dot{\theta} = \dot{\theta}_0 \cos kt. \tag{43·11}$$

The frequency of oscillation about a vertical position is

$$f = \frac{k}{2\pi} = \frac{\omega}{2\pi}\sqrt{\left\{\frac{2}{3}\left(4 - \frac{g}{r\omega^2}\right)\right\}}, \tag{43·12}$$

and thus varies practically directly as the spin velocity.

To find ϕ we substitute into (43·5) the value of θ from (43·10), giving

$$A\frac{d\phi}{dt} = C\omega\left[\frac{\dot{\theta}_0}{k} \sin kt\right].$$

Integrating this, we get

$$\phi = -\frac{C\omega\dot{\theta}_0}{A} \cos kt + C_4.$$

To find C_4, put $\phi=0$ when $t=0$. Then

$$C_4=\frac{C\omega\dot{\theta}_0}{A}.$$

Hence we get

$$\phi=\frac{C\omega\dot{\theta}_0}{A}\left(1-\cos\sqrt{\left\{\frac{2}{3}\left(4\omega^2-\frac{g}{r}\right)\right\}}t\right). \tag{43·13}$$

For a rolling *disk* or *coin* we have

$$C=\tfrac{1}{2}mr^2, \quad A=\tfrac{1}{4}mr^2, \quad I_{MN}=mr^2+\tfrac{1}{4}mr^2=\tfrac{5}{4}mr^2.$$

Substituting these into (43·6) and simplifying, we get

$$\frac{d^2\theta}{dt^2}+\frac{4}{5}\left(3\omega^2-\frac{g}{r}\right)\theta=0. \tag{43·14}$$

Here the auxiliary equation is

$$\alpha^2=-\frac{4}{5}\left(3\omega^2-\frac{g}{r}\right).$$

If the values of α are to be imaginary, we must have

$$3\omega^2>\frac{g}{r},$$

or

$$\omega^2>\frac{g}{3r}. \tag{43·15}$$

On comparing this value of ω^2 with that given by (43·9), we see that the spin velocity of a disk or coin must be greater than for a hoop if they are both to roll on without toppling over. This result agrees with everyday observation.

44. Gyroscopic grinding mills

The principle of gyroscopic reaction has been applied in two types of grinding mills where intense pressure is desired. The two types are known as the edge mill and the Griffin or pendulum mill. Both types grind by crushing.

(*a*) *Edge Mills*. The edge mill is a gyroscopic grinder used for crushing ore, seeds, etc. It consists of a large steel pan in which heavy rollers, called *mullers*, roll on the bottom of the pan and at the same time revolve about a vertical shaft passing through the

center of the pan. The mullers (usually two placed symmetrically about the shaft, but sometimes only one) are in the form of heavy flywheels with thick rims. They rotate about axles (either horizontal or inclined) which are attached to the vertical shaft by means of short cranks that permit the mullers to have enough vertical motion

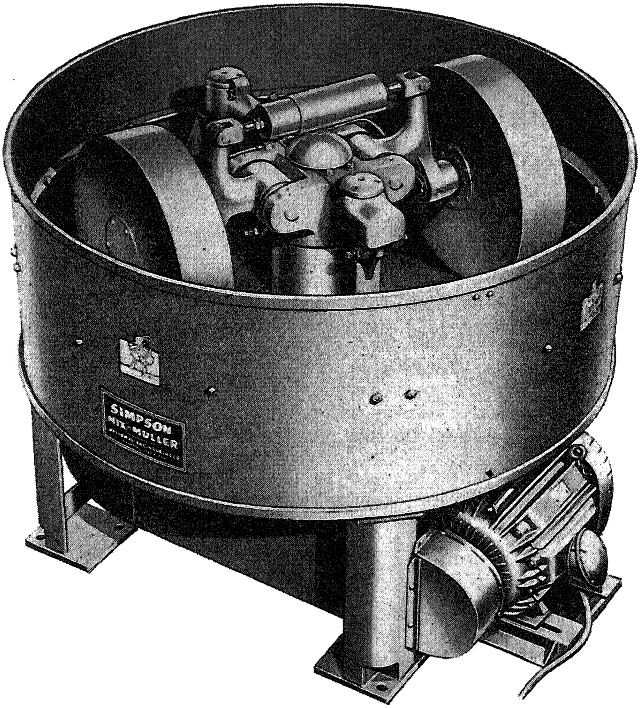


Fig. 38
(*Photograph courtesy National Engineering Co.*)

to roll over uneven lumps in the pan. The material to be ground by crushing is deposited on the bottom of the pan, through a hopper, and the mullers are forced to roll over it without slipping. A kind of plow is rigidly attached to the vertical shaft and keeps the material raked into the path of the mullers (see Fig. 38).

In the following investigation of the gyroscopic action in an edge mill,* we consider the general case in which the axle of the muller

* The gyroscopic theory of edge mills was first given by R. Grammel in 1917.

is inclined at any angle to the vertical shaft. Since the results for a single muller apply equally to the case of two mullers, we confine our discussion to the case of one muller.

Fig. 39 represents the general case of one muller. Since the muller axle is not horizontal, the face of the muller must be a frustum of a right circular cone in order to rest flatly on the bottom of the pan.

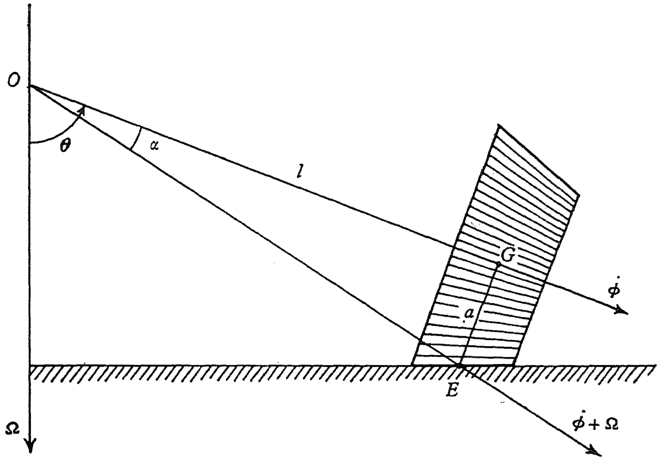


Fig. 39

If the vertical shaft is rotated at constant speed and the muller rolls without slipping, the gyroscopic reaction of the muller on the bottom of the pan is given by equation (27·1), in which A now represents the moment of inertia of the muller about an axis perpendicular to OG and passing through the fixed point O. Since by hypothesis the muller does not slip on the plate (bottom of the pan), the instantaneous axis of rotation is the line passing through the fixed point O on the vertical axis and the point E, which may be taken as the point of contact of the muller on the plate. Then if θ is the angle between the axle and the downward vertical, counted positive *from* the downward vertical *to* the axle, and α is the angle subtended at O by the mean radius GE of the muller, we have the vector parallelogram as shown in Fig. 40.

Applying the law of sines to this figure, we get

$$\frac{\dot{\phi}}{\sin(\theta-\alpha)}=\frac{\Omega}{\sin\alpha}. \tag{44·1}$$

Since rotation about the vertical axis is to be considered the independent motion in this case, we eliminate $\dot{\phi}$ between (44·1) and (27·1). Thus, from (44·1) we have

$$\dot{\phi} = \Omega \frac{\sin(\theta-\alpha)}{\sin\alpha}. \tag{44·2}$$

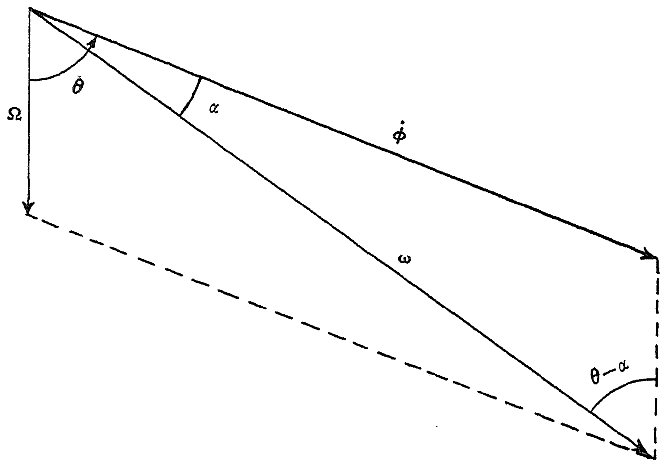


Fig. 40

Substituting this into (27·1) and disregarding the negative sign, we have

$$K = \left[C\Omega \frac{\sin(\theta-\alpha)}{\sin\alpha} + (C-A)\,\Omega\cos\theta \right] \Omega\sin\theta.$$

Expanding the factor $\sin(\theta-\alpha)$ and simplifying the result, we get

$$K = \Omega^2[C\cot\alpha\sin^2\theta - \tfrac{1}{2}A\sin 2\theta]. \tag{44·3}$$

If the muller is rotating to the right as viewed from O, it will roll to the right (clockwise) around the vertical axis as viewed from O. Hence the vectors representing $\dot{\phi}$ and Ω will be drawn as shown in Fig. 39. Then since the gyroscopic reaction moment tends to turn the spin vector into the direction of the precession vector, the gyroscopic reaction exerts a downward pressure on the grinding plate. This reaction moment is thus in the same direction as the moment due to the weight of the muller. Since the moment due to the weight of the muller is $Wl\sin\theta$, the total crushing moment is

$$M = Wl\sin\theta + \Omega^2(C\cot\alpha\sin^2\theta - \tfrac{1}{2}A\sin 2\theta). \tag{44·4}$$

On replacing $\sin^2\theta$ by $\dfrac{1-\cos 2\theta}{2}$, we have

$$M = Wl\sin\theta - \frac{\Omega^2}{2}(A\sin 2\theta + C\cot\alpha\cos 2\theta) + \frac{C\Omega^2\cot\alpha}{2}.$$

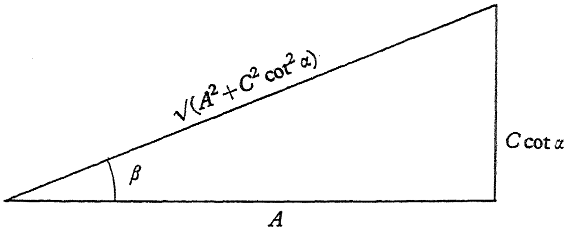


Fig. 41

The terms in parentheses can be telescoped into a single term by the well-known procedure of multiplying and dividing by the square root of the sum of the squares of the coefficients of $\sin 2\theta$ and $\cos 2\theta$. This procedure gives

$$M = Wl\sin\theta - \frac{\Omega^2}{2}\sqrt{(A^2 + C^2\cot^2\alpha)}\left(\frac{A}{\sqrt{(A^2 + C^2\cot^2\alpha)}}\sin 2\theta + \frac{C\cot\alpha}{\sqrt{(A^2 + C^2\cot^2\alpha)}}\cos 2\theta\right) + \frac{C\Omega^2\cot\alpha}{2},$$

or

$$M = Wl\sin\theta - \frac{\Omega^2}{2}\sqrt{(A^2 + C^2\cot^2\alpha)}\sin(2\theta+\beta) + \frac{C\Omega^2}{2}\cot\alpha, \tag{44·5}$$

where $\tan\beta = (C/A)\cot\alpha$ (Fig. 41). Since $\tan\beta$ is necessarily positive, β must be less than 90°.

In order to show just how the total moment M varies with the angle θ, we construct the graph of equation (44·5). That equation shows that the value of M for any given value of θ is the algebraic sum of the three quantities:

(a) $M_1 = Wl\sin\theta$,

(b) $M_2 = -\tfrac{1}{2}\Omega^2\sqrt{(A^2 + C^2\cot^2\alpha)}\sin(2\theta+\beta)$,

(c) $M_3 = \tfrac{1}{2}C\Omega^2\cot\alpha$.

Hence if we construct the graphs of (*a*), (*b*) and (*c*), we can obtain the ordinate M of (44·5) for any value of θ by adding the ordinates of (*a*), (*b*) and (*c*) for that value of θ. The period of (*a*) is 2π and that of (*b*) is π. Since M_3 is a constant, its graph is simply a straight line parallel to the θ-axis.

When $W, l, \Omega, A, C, \alpha$ and β are literal quantities, it is not possible to construct the composite graph of (44·5) without additional information. We need the values of M and $dM/d\theta$ for certain values of θ. Consequently, we next find $dM/d\theta$ and then make a table of values of M and dM/dt at 45° intervals from $\theta = 0°$ to $\theta = 180°$. Also $dM/d\theta$ for $\theta = 135° - \frac{1}{2}\beta$.

From (44·4) and (44·5) we get

$$\left.\begin{aligned} \frac{dM}{d\theta} &= Wl\cos\theta + \Omega^2(C\cot\alpha\sin 2\theta - A\cos 2\theta), \qquad \text{from (44·4)},\\ &= Wl\cos\theta - \Omega^2\sqrt{(A^2 + C^2\cot^2\alpha)}\cos(2\theta + \beta), \quad \text{from (44·5)}. \end{aligned}\right\} \tag{44·6}$$

Then from (44·4) and (44·6) we get Table 1. A glance at this table shows that the slope of the graph is positive for $\theta = 45°$ and $\theta = 90°$ and that it is negative for $\theta = 135° - \frac{1}{2}\beta$, $\theta = 135°$ and $\theta = 180°$, whatever be the values of the constants involved. Hence the maximum value of M occurs between $\theta = 90°$ and $\theta = 135° - \frac{1}{2}\beta$. Having this information and the graphs of (*a*), (*b*) and (*c*), we construct the graph for M as indicated by (5) in Fig. 42.

TABLE 1

θ	M	$dM/d\theta$
0°	0	$Wl - A\Omega^2$
45°	$\frac{1}{\sqrt{2}}Wl + \frac{1}{2}C\Omega^2\cot\alpha - \frac{1}{2}A\Omega^2$	$\frac{1}{\sqrt{2}}Wl + C\Omega^2\cot\alpha$
90°	$Wl + C\Omega^2\cot\alpha$	$A\Omega^2$
$135° - \frac{1}{2}\beta$	—	$Wl\cos(135° - \frac{1}{2}\beta)$, negative
135°	$\frac{1}{\sqrt{2}}Wl + \frac{1}{2}C\Omega^2\cot\alpha + \frac{1}{2}A\Omega^2$	$-\frac{1}{\sqrt{2}}Wl - C\Omega^2\cot\alpha$
180°	0	$-Wl - A\Omega^2$

Curve (3) is the graph of (44·3) and shows that in the interval $\theta = 0°$ to $\theta = 75°$ (approximately) the gyroscopic moment is negative and actually tends to lift the muller from the pan. The total moment is also negative in the interval $\theta = 0°$ to $\theta = 55°$ (approximately).

Note. There are crushing machines similar to that of Fig. 38 in which the pan rotates under the mullers, causing them to rotate on

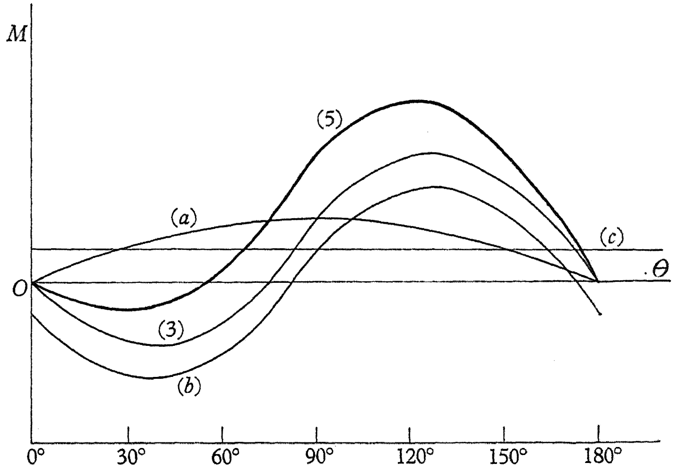


Fig. 42

their axles but not revolve about the vertical axis. In such machines there is no gyroscopic action whatever, the crushing being due entirely to the weight of the mullers. The crushing pressure is therefore much less than that in the gyroscopic type of machine.

Numerical Example. Let us consider an edge mill having a single muller weighing 2200 pounds and with a mean radius of 1.5 feet, and let us assume that the muller rotates about the vertical shaft at 40 r.p.m. Then $\Omega = (40 \times 2\pi)/60 = \frac{4}{3}\pi$ radians per second. If

$$OG = l = 2.5 \text{ feet,}$$

then
$$\tan\alpha = \frac{1.5}{2.5} = 0.6,$$

and
$$\alpha = 30° \, 58' = 31°, \quad \text{say.}$$

Taking the radius of gyration of the muller about its axle to be $0.8r = 0.8 \times 1.5 = 1.2$ feet, we have

$$C = \frac{2200}{32.2}(1.2)^2 = 98.4,$$

$$A = \frac{2200}{32.2}(2.5)^2 + \tfrac{1}{2}(98.4) \quad \text{approximately}$$

$$= 476 \text{ approximately.}$$

Then $$\tan\beta = \frac{C\cot\alpha}{A} = \frac{98.4\cot 31^\circ}{476} = 0.345,$$

$$\beta = 19^\circ\,02' = 19^\circ, \quad \text{say.}$$

We next find the value of θ for maximum pressure moment M. Putting $dM/d\theta = 0$ in the second of equations (44·6), we have

$$Wl\cos\theta - \Omega^2\sqrt{(A^2 + C^2\cot^2\alpha)}\cos(2\theta + \beta) = 0.$$

On substituting the above numerical values for the constants in this equation, we get

$$5500\cos\theta - \left(\frac{4\pi}{3}\right)^2\sqrt{\{(476)^2 + (98.4)^2(1.666)^2\}}\cos(2\theta + 19^\circ) = 0,$$

which reduces to

$$\cos(2\theta + 19^\circ) - 0.622\cos\theta = 0.$$

This equation must be solved by trial, and it will be found that $\theta = 117^\circ\,14'$ satisfies it.

Substituting into (44·5) this value of θ and the numerical values of the constants as given above, we have

$$M = 5500\sin 117^\circ - 8.78 \times 503\sin 253^\circ + 8.78 \times 164$$

$$= 4900 + 5660 = 10{,}560 \text{ lb.ft.}$$

The gyroscopic moment is thus greater than the moment due to the weight of the muller.

For $\theta = 90^\circ$ we would have

$$M = 5500 + 2880 = 8380\,\text{lb.ft.}$$

In this case the gyroscopic moment is only slightly more than half as great as before. But in the case of a grinder such as that pictured

in Fig. 38, for example, the G.R.M. would still increase the pressure on the pan by more than 50 % above that due to the weight of the rollers.

Equation (44·3) shows that the gyroscopic moment varies as the square of the speed of revolution about the vertical shaft.

The rotational speed of the muller about its axle is found from equation (44·2) to be

$$\dot{\phi}=40\frac{\sin 86^\circ}{\sin 31^\circ}=77.5\,\text{r.p.m.}$$

(*b*) *The Griffin Grinding Mill.* Although this mill grinds by intense pressure due entirely to gyroscopic action, it is quite different in construction from the edge mill. In the Griffin mill the grinding is done by a pendulous cylinder attached to a vertical shaft by means of a Hooke or universal joint. The rotating cylinder presses against the inner wall of a heavy steel pan as it rolls around inside the pan, as indicated in Fig. 43. This grinder is also called a pendulum mill because of the pendulous roller it employs.

To derive the formula for the crushing moment in this mill, we again apply equation (27·1). In this grinder the rotational speed of the vertical shaft above the Hooke joint is the same as the speed ω about the instantaneous axis, as can be seen by imagining the joint replaced by a flexible shaft. Furthermore, a moment's reflection or a simple experiment will show that when the cylinder rotates to the right (clockwise) and rolls around the *inner* wall of the pan without slipping, it will move around the central vertical axis in a counterclockwise direction as viewed from the fixed point (Hooke's joint) above. In other words, the axle of the spinning roller precesses to the left. This means that $\dot{\phi}$ and Ω have opposite signs and that the spin ω of the vertical shaft is the algebraic sum of these speeds of opposite sign. Hence we have

$$\omega=\dot{\phi}+\Omega. \tag{44·7}$$

Since the speed of the vertical shaft is the controlling motion and known speed in this grinder, we must express $\dot{\phi}$ and Ω in terms of ω before using (27·1).

From equation (44·1) we have

$$\frac{\Omega}{\dot{\phi}}=\frac{\sin\alpha}{\sin(\theta-\alpha)}. \tag{44·8}$$

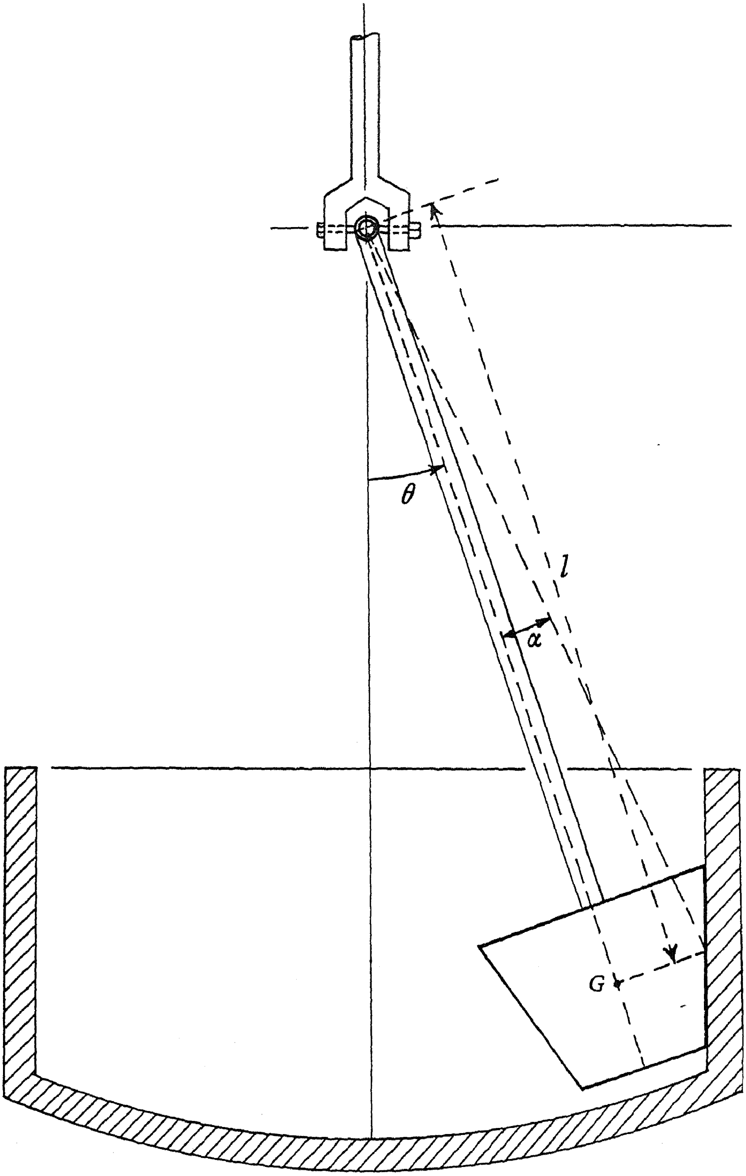


Fig. 43

Adding 1 to each side of this equation, we have

$$\frac{\Omega}{\dot\phi}+1=\frac{\sin\alpha}{\sin(\theta-\alpha)}+1,$$

from which

$$\frac{\Omega+\dot\phi}{\dot\phi}=\frac{\sin\alpha+\sin(\theta-\alpha)}{\sin(\theta-\alpha)}.$$

But $\Omega+\dot\phi=\omega$, by (44·7). Hence

$$\frac{\omega}{\dot\phi}=\frac{\sin\alpha+\sin(\theta-\alpha)}{\sin(\theta-\alpha)}.$$

$$\therefore\quad \dot\phi=\frac{\omega\sin(\theta-\alpha)}{\sin\alpha+\sin(\theta-\alpha)}. \tag{44·9}$$

Now replacing $\dot\phi$ in (44·1) by its value as given by (44·9), we get

$$\Omega=\frac{\omega\sin\alpha}{\sin\alpha+\sin(\theta-\alpha)}. \tag{44·10}$$

On replacing $\dot\phi$ and Ω in (27·1) by their values given in (44·9) and (44·10), disregarding the minus sign, and then expanding the factor $\sin(\theta-\alpha)$ in the numerator of the result and simplifying slightly, we get

$$K=\frac{\omega^2\sin\alpha\sin\theta}{[\sin\alpha+\sin(\theta-\alpha)]^2}(C\cos\alpha\sin\theta-A\sin\alpha\cos\theta) \tag{44·11}$$

for the gyroscopic reaction moment.

If the cylinder is rotating to the right (clockwise) as viewed from above, the spin vector must be drawn downward along the axis of the cylinder, and the precession vector must be drawn vertically upward. Then since the rule for the direction of the gyroscopic reaction moment requires that it tend to turn the positive end of the spin vector into the direction of the positive end of the precession vector, we see that the moment K tends to press the cylinder against the wall of the pan. Since the gravity moment due to the weight of the cylinder tends to pull the cylinder away from the wall, it is plain that the gyroscopic moment must exceed the gravity moment in order to keep the cylinder in contact with the wall. The crushing moment of the cylinder is the algebraic sum of these two moments, so that we have

$$M=Wl\sin\theta+\frac{\omega^2\sin\alpha\sin\theta}{[\sin\alpha+\sin(\theta-\alpha)]^2}(C\cos\alpha\sin\theta-A\sin\alpha\cos\theta), \tag{44·12}$$

or

$$M = Wl \sin\theta + \frac{\omega^2 \sin\alpha}{[\sin\alpha + \sin(\theta-\alpha)]^2}(C\cos\alpha\sin^2\theta - \tfrac{1}{2}A\sin\alpha\sin 2\theta). \tag{44·12a}$$

This moment, in the case of a given cylinder, depends on ω and θ.

In determining the sign and magnitude of θ we must keep in mind the physical conditions of the problem:

(1) $\dot{\phi}$ and Ω must be of opposite sign.

(2) $\dot{\phi} > \Omega$, because the circumference of the pan is necessarily greater than that of the cylinder which rolls inside it.

(3) $A \gg C$ in this problem.

(4) The gravity moment $Wl\sin\theta$ and the gyroscopic moment K must be of opposite sign.

Equations (44·8) and (44·12a) show that these conditions will be fulfilled only when θ is a negative acute angle, or $-90° < \theta < 0°$. There is no best value of θ in this case as was true with the edge mill. The gyroscopic moment continues to increase as θ decreases in magnitude.

Numerical Example. The muller of a Griffin grinding mill is in the form of a frustum of a right circular cone and weighs 980 pounds. The altitude of the frustum is 1 foot, the radius of the upper base is 1.077 feet, and the radius of the lower base is 0.5 foot (Fig. 43). (The generating line of the conical surface makes a 30° angle with the geometric axis of the frustum.)

The center of gravity of the muller is found to be 0.383 foot from the upper base. Taking $l = 3.383$ feet and the radius through the center of gravity as 0.856 foot, we have

$$\tan\alpha = \frac{0.856}{3.383} = 0{\cdot}253,$$

$$\alpha = 14°.2.$$

Further computation gives

$$C = 11.52,$$

$$A = \frac{980}{32.2}(3.383)^2 + 7.98 = 356.$$

Assuming that the vertical shaft (the part above the Hooke joint) rotates at 240 r.p.m., we have

$$\omega = \frac{240}{60} \times 2\pi = 8\pi \text{ radians per second.}$$

Taking $\theta = -30°$ and substituting these data into (44·12) or (44·12a), we find

$$M = -1658 + 30810 = 29{,}150 \text{ lb.ft.}$$

Formulas (44·9) and (44·10) give

$$\dot{\phi} = 38.75 \text{ radians per second} = 370 \text{ r.p.m.},$$

$$\Omega = -13.65 \text{ radians per second} = -130.3 \text{ r.p.m.}$$

To find the angular velocity ω necessary to keep the muller in contact with the wall of the pan, we must have

$$K > Wl \sin 30°,$$

or

$$\omega^2\left(\frac{0.1226}{0.2043}\right)(81.2) > 1658.$$

Hence $\omega > 5.83$ radians per second $= 55.7$ r.p.m.

45. Gyroscopic action on oblong projectiles fired from rifled guns

A rifled gun barrel is one having spiral grooves on its interior surface or bore. Such grooves impart a rotary motion to projectiles fired from the gun. The pitch or twist of the grooves is such as to give the projectile a high rotational spin about its geometric axis.

When an oblong projectile is fired from a gun having a right-hand twist, it swerves slightly to the right of the plane of fire (the vertical plane containing the axis of the gun barrel), as viewed from the rear of the gun; and when such a projectile is fired from a gun having left-hand twist, it swerves slightly to the left of the plane of fire. This swerving to the right or to the left of the plane of fire is called *drift* and is due mostly to gyroscopic action.

Consider an oblong projectile fired from a rifled gun having right-hand twist. At the instant when the projectile leaves the gun, its axis coincides with the axis of the barrel and is therefore tangent to the trajectory (path of the center of gravity of the projectile) at that instant. Immediately after the projectile leaves the gun, however, its center of gravity is pulled downward continuously by the force of gravity, thus causing the center of gravity to trace a curved path in a vertical plane. Since the high rotational spin tends to keep the axis of spin oriented in its original direction in space, and gravity is all the while changing the direction of the tangent to

the path of the c.g., it is plain that the axis of the projectile will soon make an angle α with the tangent, as indicated in Fig. 44.

The resultant air resistance opposing the motion of the projectile is practically parallel to the tangent and always intersects the axis of spin at some point A in front of the c.g. of the projectile. In Fig. 44, O designates the c.g. of the projectile, OT the tangent to the trajectory, and R the air resistance.

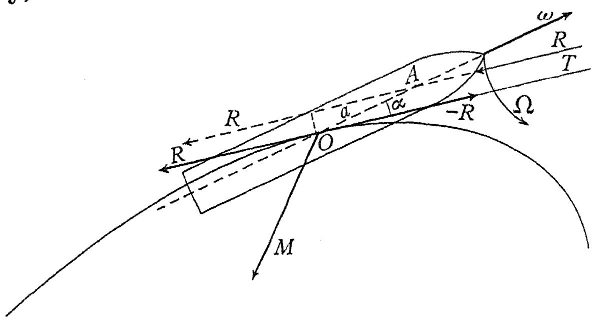


Fig. 44

The force R can be replaced by a force R acting at O and a couple R, $-R$ as shown. The moment of the couple is $Ra \sin \alpha$, where $a = OA$, and tends to rotate the projectile about a transverse axis passing through O, this axis being normal to the vertical plane containing OT. This moment is represented by the vector **M** and must be drawn to the right (as viewed from the rear) of the vertical plane through OT in order to conform to the right-handed-screw convention.

Let ω denote the angular velocity of the projectile about its geometric axis. Then the moment **M** will cause the point of the projectile to precess with angular velocity Ω about a line through O and drawn in the direction of the resultant air resistance, which line is practically OT. By formula (22·3) the rate of precession is

$$\dot{\psi} = \frac{M}{C\omega \sin \alpha} = \frac{Ra \sin \alpha}{C\omega \sin \alpha} = \frac{Ra}{C\omega},$$

where C denotes the moment of inertia of the projectile about its axis of spin.

It might seem from the above considerations that the point of the projectile would describe a circular helix about the trajectory, but this is not the case. It can be shown both graphically and

analytically that the point of the projectile describes a cycloidal space curve which always lies on the right-hand side of the vertical plane through OT. The proof of this fact lies properly in advanced works on exterior ballistics and will therefore not be given here.*

The fact that the axis of the projectile is always pointing to the right of the vertical plane through the tangent shows that the air force (which is parallel to the tangent) has a small rectangular component which is perpendicular to the axis of the projectile and acts toward the right. It is this small force which causes the projectile to swerve to the right.

46. Gyroscopic effects on propeller-type airplanes

An airplane propeller spinning at high speed has the dynamical properties of a spinning gyro. In the case of an airplane equipped with a single propeller a turn in azimuth, for example, is a forced precession of the spin axis about a vertical axis. If the propeller is rotating clockwise as viewed from the rear, this forced precession will induce a G.R.M. which will cause the nose of the plane to move upward or downward according as the turn in azimuth is to the left or to the right. Turning the nose upward rather suddenly will cause the plane to turn to the right, thus tending to follow an upward spiral. On the other hand, turning the nose downward rather suddenly will cause the plane to turn to the left, thus tending to follow a downward spiral.

When an airplane is equipped with two propellers rotating in opposite directions and at the same speed, the gyroscopic effect of either propeller annuls that of the other, and there is therefore no appreciable gyroscopic effect on the plane as a whole.

* The interested reader should consult the following: *New Methods in Exterior Ballistics*, ch. 6, by F. R. Moulton, 1926; *Ballistics of the Future*, ch. 8, by Kooy and Uytenbogaart, 1946; *Äussere Ballistik*, sections 57 and 58, by Cranz, Von Eberhard, and Becker, 1925.

CHAPTER VI

The Gyroscope as a Direction Indicator and Steering Device

A. THE GYROSCOPIC COMPASS

47. The principle of action

The stubborn tendency of a fast-spinning flywheel to maintain its plane of rotation early suggested the use of a gyroscope as a compass to replace the magnetic compass. If the axle of a fast-spinning free gyroscope is pointed toward a fixed star, for example, it will continue to point toward that star (because of the principle of the conservation of angular momentum) as long as the gyro is kept spinning at high speed. But if the spinning gyro is set with its axle horizontal and in the meridian plane at any point on the earth's surface, the axle will remain horizontal and in the meridian plane if it is compelled to precess about the earth's axis at the same angular velocity as the earth rotates, or precess about the vertical with an angular velocity equal to the vertical component of the earth's angular velocity at that place, thus making the change in direction of the gyro axle keep pace with the change in direction (with respect to fixed space) of the meridian at that point (see Fig. 45). This is the principle of action of the gyrocompass. Hence, if the axle of a spinning gyro is pointed toward the north point of the horizon, it will continue to point to the true north if it is compelled to precess in the right direction and at the proper rate.

The gyro axle can be made to precess at any desired rate by attaching a pendulous weight to the frame which supports the gyro axle, as we learned in Chapter IV. In order to determine on which side of the frame the weight must be attached, it is first necessary to know in what direction the axle must precess. Gyrocompasses are mounted on board ship with the gyro axle practically horizontal and the compass card (dial) facing upward. Hence the compass must

be read by looking down on it from above. A glance at Fig. 45 shows that the spin axis must precess to the left (counterclockwise) as viewed from above, in order to keep pace with the changing direction of the meridian. Consequently, the pendulous weight must be attached as indicated in Fig. 46, for in that position the moment vector of the weight is directed as indicated in the figure and the spin vector must be rotated counterclockwise (as viewed from above) in order to bring it into coincidence with the moment vector (Art. 23).

The precession of the gyro axle may be looked upon either as a precession of angular velocity Ω about the polar axis of the earth or of an angular velocity $\Omega \sin \lambda$ about a vertical axis at latitude λ (see Fig. 45). In either case we have, from Fig. 46 and (22·3) in the first case or (22·4) in the second case, the relation

$$wa \sin \beta = C\omega\Omega \sin \lambda, \tag{47·1}$$

where ω = angular velocity of spin of gyro in radians per second,

Ω = angular velocity of earth's rotation in radians per second,

C = moment of inertia of gyro about its axis of spin in pound-feet-second units,

λ = latitude of position of gyrocompass,

w = weight of pendulous weight in pounds,

a = moment arm of w in feet,

β = angle which arm a makes with vertical.

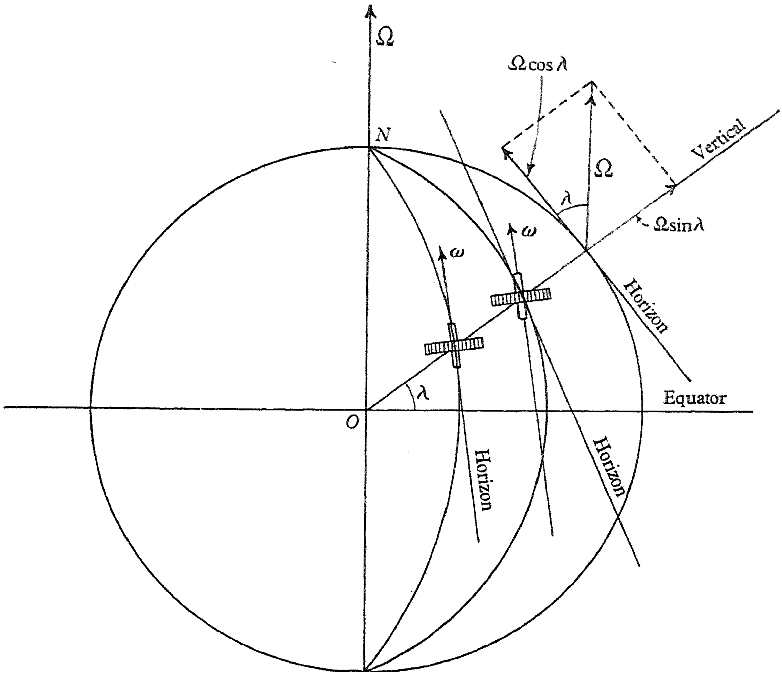
If β_0 is the value of β which is necessary to give the gyro axle the proper precession to keep it in the meridian, we have from (47·1)

$$\sin \beta_0 = \frac{C\omega\Omega \sin \lambda}{wa}.$$

But β_0, as will be seen later, is a very small angle of 5′ to 10′ in magnitude. Hence we may replace $\sin \beta_0$ by β_0. Then we have

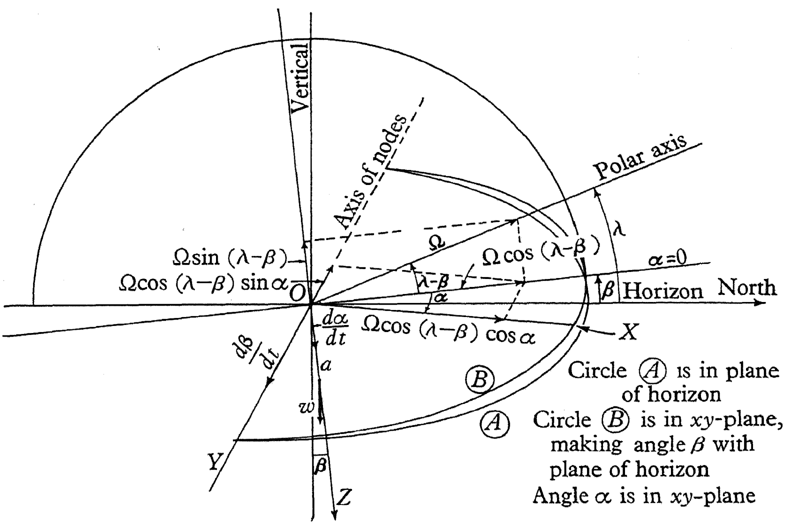
$$\beta_0 = \frac{C\omega\Omega \sin \lambda}{wa}. \tag{47·2}$$

The gyro axle is therefore not horizontal in its equilibrium position but is tilted slightly upward.



Ω
Ω cos λ
N
Vertical
Ω
λ
Ω sin λ
ω
ω
Horizon
λ
Equator
O
Horizon
Horizon

Fig. 45



Vertical
Axis of nodes
Polar axis
Ω
Ω cos (λ-β)
λ
α=0
Ωsin (λ-β)
Ωcos (λ-β) sin α
O
λ-β
α
β Horizon North
dα/dt
Ωcos (λ-β) cos α
X
dβ/dt
a
w
B
A
Y
β
Z
Circle A is in plane of horizon
Circle B is in xy-plane, making angle β with plane of horizon
Angle α is in xy-plane

Fig. 46

The angular velocity of the earth's rotation, in radians per second, is equal to 2π divided by the number of seconds in a sidereal day, or

$$\Omega = \frac{2\pi}{86164.091} = 0.000072921158 \text{ radian per second.}$$

This number, rounded to four significant figures, will be used frequently in the work that follows. Note that

$$\Omega^2 = 0.000{,}000{,}005{,}317{,}3.$$

This quantity is so small that it will henceforth be neglected wherever it occurs, except when it is to be multiplied by a very large number.

48. Differential equations of motion of a disturbed gyrocompass axle

To determine the motion of the gyro axle when disturbed from its position of equilibrium, we take a right-handed set of rectangular axes rigidly attached to the frame of the gyroscope and with origin at the center of gravity of the rotor (which is practically the center of gravity of the movable system), as indicated in Fig. 46. The x-axis coincides with the axis of spin, with positive end toward the north. The y-axis coincides with the axis of nodes (moment axis of pendulous weight), the positive end being directed toward the reader. The z-axis is perpendicular to the plane of the other two and is positive *downward*, because the compass card is graduated with 0° at the north and with the numbers increasing toward the east (clockwise as viewed from above) to 360° and is read by looking down on it. Horizontal or azimuth angles will be taken as positive from the north toward the east. Since the earth rotates from west to east (as one looks toward the north), Ω will always be taken as positive. All the above conventions agree with the right-handed-screw rule used throughout this book.

The mounting of the gyro is such that its axis of spin is free to move in any manner, but the axis of nodes (OY) is constrained to remain approximately horizontal.

Figure 46 shows that the components of the angular velocity of the earth and of the frame along the coordinate axes are as follows:

$$\left.\begin{aligned} \omega_x &= \Omega \cos(\lambda - \beta) \cos\alpha, \\ \omega_y &= \frac{d\beta}{dt} - \Omega \cos(\lambda - \beta) \sin\alpha, \\ \omega_z &= \frac{d\alpha}{dt} - \Omega \sin(\lambda - \beta). \end{aligned}\right\} \qquad (48{\cdot}1)$$

Hence the components of the angular momentum on the axes are

$$\left.\begin{aligned} H_x &= C\Omega\cos(\lambda-\beta)\cos\alpha + C\omega, \\ H_y &= A\frac{d\beta}{dt} - A\Omega\cos(\lambda-\beta)\sin\alpha, \\ H_z &= A\frac{d\alpha}{dt} - A\Omega\sin(\lambda-\beta), \end{aligned}\right\} \tag{48·2}$$

where A denotes the moment of inertia of the gyro and frame about an axis through the center of gravity of the gyro and perpendicular to the axis of spin. Hence we have

$$\frac{dH_x}{dt} = C\Omega\left[\cos\alpha\sin(\lambda-\beta)\frac{d\beta}{dt} - \sin\alpha\cos(\lambda-\beta)\frac{d\alpha}{dt}\right],$$

$$\frac{dH_y}{dt} = A\frac{d^2\beta}{dt^2} - A\Omega\left[\sin\alpha\sin(\lambda-\beta)\frac{d\beta}{dt} + \cos\alpha\cos(\lambda-\beta)\frac{d\alpha}{dt}\right],$$

$$\frac{dH_z}{dt} = A\frac{d^2\alpha}{dt^2} + A\Omega\cos(\lambda-\beta)\frac{d\beta}{dt}.$$

It will be seen later that $d\alpha/dt$ and $d\beta/dt$ are very small. Hence the products $\Omega(d\alpha/dt)$ and $\Omega(d\beta/dt)$ are negligible. Neglecting such products in the above derivatives, we get

$$\frac{dH_x}{dt} = 0, \quad \frac{dH_y}{dt} = A\frac{d^2\beta}{dt^2}, \quad \frac{dH_z}{dt} = A\frac{d^2\alpha}{dt^2}.$$

On substituting into (16·4) these derivatives and the quantities in (48·1) and (48·2), then neglecting all terms containing the negligible quantities Ω^2, $\Omega(d\alpha/dt)$, $\Omega(d\beta/dt)$, and $(d\alpha/dt)(d\beta/dt)$, we get

$$\left.\begin{aligned} A\frac{d^2\beta}{dt^2} + C\omega\frac{d\alpha}{dt} - C\omega\Omega\sin(\lambda-\beta) &= -wa\sin\beta, \\ A\frac{d^2\alpha}{dt^2} - C\omega\frac{d\beta}{dt} + C\omega\Omega\sin\alpha\cos(\lambda-\beta) &= 0; \end{aligned}\right\} \tag{48·3}$$

or, after expanding $\sin(\lambda-\beta)$ and $\cos(\lambda-\beta)$:

$$A\frac{d^2\beta}{dt^2} + C\omega\frac{d\alpha}{dt} - C\omega\Omega\sin\lambda\cos\beta + C\omega\Omega\cos\lambda\sin\beta = -wa\sin\beta,$$

$$A\frac{d^2\alpha}{dt^2} - C\omega\frac{d\beta}{dt} + C\omega\Omega\sin\alpha\cos\lambda\cos\beta + C\omega\Omega\sin\alpha\sin\lambda\sin\beta = 0.$$

Now since β is a very small angle, we may neglect the terms containing the product $\Omega \sin \beta$. We may also put $\cos \beta = 1$ and $\sin \beta = \beta$. Then the above equations become

$$\left.\begin{aligned} A\frac{d^2\beta}{dt^2} + C\omega\frac{d\alpha}{dt} - C\omega\Omega\sin\lambda + wa\beta = 0, \\ A\frac{d^2\alpha}{dt^2} - C\omega\frac{d\beta}{dt} + C\omega\Omega\cos\lambda\sin\alpha = 0. \end{aligned}\right\} \tag{48·4}$$

These are the fundamental differential equations which determine the motion of the gyro axle when disturbed from its position of equilibrium. Since the quantities $d^2\alpha/dt^2$, $d\alpha/dt$, $d^2\beta/dt^2$ and $d\beta/dt$ are all zero in the equilibrium position, we can find that position by putting these quantities equal to zero in equations (48·4). We thus get

$$-C\omega\Omega\sin\lambda + wa\beta_0 = 0,$$

$$C\omega\Omega\cos\lambda\sin\alpha = 0.$$

Hence $$\alpha = 0$$

and $$\beta_0 = \frac{C\omega\Omega\sin\lambda}{wa}.$$

This value of β_0 is the same as already found in (47·2). Hence in the equilibrium position the gyro axle is in the meridian plane and its north end elevated at the small angle β_0 above the horizon.

Equations (48·4) can be reduced to a symmetrical form by making the substitution

$$\chi = \beta - \beta_0 = \beta - \frac{C\omega\Omega\sin\lambda}{wa}, \tag{48·5}$$

or $$\beta = \beta_0 + \chi = \chi + \frac{C\omega\Omega\sin\lambda}{wa}.$$

Hence $$\frac{d\beta}{dt} = \frac{d\chi}{dt}, \quad \frac{d^2\beta}{dt^2} = \frac{d^2\chi}{dt^2}.$$

Making these substitutions into (48·4) and putting $\sin\alpha = \alpha$, since α is a small angle, we get

$$\left.\begin{aligned} (a)\ \ A\frac{d^2\chi}{dt^2} + C\omega\frac{d\alpha}{dt} + wa\chi = 0, \\ (b)\ \ A\frac{d^2\alpha}{dt^2} - C\omega\frac{d\chi}{dt} + C\omega\Omega\cos\lambda\,.\,\alpha = 0. \end{aligned}\right\} \tag{48·6}$$

49. Derivation of the differential equations by the G.R.M. method

Equations (48·4) can also be derived by considering all the moments, external and G.R.M., about the axes OY and OZ.

Let us first consider the moments about OY. The moment of the pendulous weight is $-wa\sin\beta$, negative because it tends to decrease β. The precession $-\Omega\sin(\lambda-\beta)$ about OZ induces a G.R.M. $-[C\omega(-\Omega\sin(\lambda-\beta))]$, or $C\omega\Omega\sin(\lambda-\beta)$, about OY. Also, the angular velocity $d\alpha/dt$ induces a G.R.M. $-C\omega\, d\alpha/dt$ about OY. Hence the total moment about OY is

$$-wa\sin\beta+C\omega\Omega\sin(\lambda-\beta)-C\omega\frac{d\alpha}{dt}.$$

Then by the fundamental equation of rotation (20·2) we have

$$A\frac{d^2\beta}{dt^2}=C\omega\Omega\sin(\lambda-\beta)-C\omega\frac{d\alpha}{dt}-wa\sin\beta,$$

or
$$A\frac{d^2\beta}{dt^2}+C\omega\frac{d\alpha}{dt}-C\omega\Omega\sin(\lambda-\beta)+wa\sin\beta=0,$$

which is the first of equations (48·3).

Now considering the moments about OZ, we note first of all that there is no external moment. The forced rotation (precession) $d\beta/dt$ induces a G.R.M. $C\omega(d\beta/dt)$ about OZ. Also, the precession $-\Omega\cos(\lambda-\beta)\sin\alpha$ about OY induces a G.R.M.

$$C\omega[-\Omega\cos(\lambda-\beta)\sin\alpha]$$

about OZ. Hence by the rotational equation (20·2) we have

$$A\frac{d^2\alpha}{dt^2}=C\omega\frac{d\beta}{dt}-C\omega\Omega\cos(\lambda-\beta)\sin\alpha,$$

or
$$A\frac{d^2\alpha}{dt^2}-C\omega\frac{d\beta}{dt}+C\omega\Omega\cos(\lambda-\beta)\sin\alpha=0,$$

which is the second of equations (48·3).

Although the G.R.M. method is much shorter than the use of equations (16·4), it is less foolproof. The student is advised to write down the angular velocities ω_x, ω_y, ω_z on the three axes and use equations (16·4) until he becomes expert in the use of the G.R.M. method.

50. Solution of the equations

The simultaneous equations (48·6) can be solved by eliminating one of the variables and thereby obtaining an equation of the fourth order in the other variable. The elimination can be effected in several ways. A simple and direct method of eliminating χ, for example, is the following:

Differentiate (a) with respect to t and then substitute into the result the value of $d\chi/dt$ obtained from (b), thus obtaining

$$(c)\quad A\frac{d^3\chi}{dt^3}+C\omega\frac{d^2\alpha}{dt^2}+\frac{Awa}{C\omega}\frac{d^2\alpha}{dt^2}+\Omega wa\cos\lambda\,.\,\alpha=0.$$

Now differentiate (b) twice with respect to t, obtaining

$$(d)\quad A\frac{d^4\alpha}{dt^4}-C\omega\frac{d^3\chi}{dt^3}+C\omega\Omega\cos\lambda\frac{d^2\alpha}{dt^2}=0.$$

Eliminating $d^3\chi/dt^3$ between (c) and (d) and then simplifying the result slightly, we get

$$A^2\frac{d^4\alpha}{dt^4}+(C^2\omega^2+AC\omega\Omega\cos\lambda+Awa)\frac{d^2\alpha}{dt^2}+C\omega\Omega wa\cos\lambda\,.\,\alpha=0. \tag{50·1}$$

By following a similar procedure we can eliminate α from equations (48·6) and get

$$A^2\frac{d^4\chi}{dt^4}+(C^2\omega^2+AC\omega\Omega\cos\lambda+Awa)\frac{d^2\chi}{dt^2}+C\omega\Omega wa\cos\lambda\,.\,\chi=0. \tag{50·2}$$

This equation is exactly the same as (50·1) except that α has been replaced by χ. Hence the motion of the gyro axle in the vertical direction is of exactly the same character as its motion in the horizontal direction (in or parallel to the plane $\beta=\beta_0$).

The auxiliary equation for either (50·1) or (50·2) is

$$A^2r^4+(C^2\omega^2+AC\omega\Omega\cos\lambda+Awa)\,r^2+C\omega\Omega wa\cos\lambda=0.$$

Hence by the quadratic formula we have

$$r^2=\frac{-(C^2\omega^2+AC\omega\Omega\cos\lambda+Awa)\pm\sqrt{\{(C^2\omega^2+AC\omega\Omega\cos\lambda+Awa)^2-4A^2C\omega\Omega wa\cos\lambda\}}}{2A^2}.$$

Now since $(C^2\omega^2+AC\omega\Omega\cos\lambda+Awa)^2$ is very much greater than $4A^2C\omega\Omega wa\cos\lambda$, the radicand is positive. Consequently, the two

values of r^2 are real. Moreover, they are both negative, because the radical is less than $C^2\omega^2 + AC\omega\Omega\cos\lambda + Awa$.

Let these negative values of r^2 be $-k_1^2$ and $-k_2^2$, so that

$$r_1^2 = -k_1^2, \quad r_2^2 = -k_2^2.$$

Then $$r_1 = \pm ik_1, \quad r_2 = \pm ik_2.$$

Hence the general solution for α is

$$\alpha = c_1 \sin k_1 t + c_2 \cos k_1 t + c_3 \sin k_2 t + c_4 \cos k_2 t.$$

A similar solution holds for χ.

The above solution represents two simultaneous and superposed simple harmonic motions of periods $2\pi/k_1$ and $2\pi/k_2$, respectively.

51. A numerical example

At this point it is instructive to work out a numerical example in order to obtain an idea of the magnitudes of the several terms in (50·1). The following data are approximately correct for some of the later Sperry gyrocompasses:

$$\omega = 8500 \text{ r.p.m.} = 890 \text{ radians per second,}$$
$$C = 0.25 \text{ in foot-pound-second units,}$$
$$A = 0.15 \text{ in foot-pound-second units,}$$
$$wa = 6.5 \text{ pound feet.}$$

Hence $$C\omega = 222.5.$$

Also $$\Omega = 0{\cdot}00007292 \text{ radian per second.}$$

Take $$\lambda = 41^\circ \text{ N.}$$

On substituting these quantities into (50·1) we have

$$0.0225\frac{d^4\alpha}{dt^4} + (49506.25 + 0.0018370 + 0.975)\frac{d^2\alpha}{dt^2} + 0.079592\,\alpha = 0,$$

or $$0{\cdot}0225\frac{d^4\alpha}{dt^4} + 49507.226837\frac{d^2\alpha}{dt^2} + 0.079592\,\alpha = 0.$$

Hence $$0.0225\,r^4 + 49507.226837\,r^2 + 0.079592 = 0,$$

and therefore

$$r^2 = \frac{-49507.226837 \pm \sqrt{\{(49507.226837)^2 - 0.00716328\}}}{0.045},$$

$$r_1^2 = \frac{-49507.226837}{0.045}\left\{1 \mp \left[1 - \frac{0.00716328}{(49507.226837)^2}\right]^{\frac{1}{2}}\right\}.$$

Expanding the quantity in square brackets into a binomial series by means of the formula

$$(1+u)^n = 1 + nu + \frac{n(n-1)}{2}u^2 + \ldots$$

and retaining only the first two terms of the expansion, we get

$$r^2 = \frac{-49507.226837}{0.045}\left\{1 \mp \left[1 - \frac{0.00358164}{(49507.226837)^2}\right]\right\}.$$

Hence

$$r_1^2 = \frac{-49507.226837}{0.045}\left\{1 - 1 + \frac{0.00358164}{(49507.226837)^2}\right\}$$

$$= \frac{-0.00358164}{0.045 \times 49507{\cdot}226837} = -0.0000016076845.$$

$$\therefore \quad r_1 = \pm 0.00126795i.$$

The period of oscillation is therefore

$$T_1 = \frac{2\pi}{0.00126795} = 4955{\cdot}4 \text{ seconds}$$

$$= 82{\cdot}6 \text{ minutes}.$$

For the other two roots we have

$$r_2^2 = \frac{49507{\cdot}226837}{-0.045}\left\{1 + 1 - \frac{0.00358164}{(49507.226837)^2}\right\}$$

$$= -\frac{99014.453674}{0.045} + \frac{0.00358164}{0{\cdot}045 \times 49507.226837}$$

$$= -2200321.192754.$$

Hence $$r_2 = \pm 1483.348i,$$

and the period in this case is

$$T_2 = \frac{2\pi}{1483.348} = 0.0042 \text{ second}.$$

This exceedingly short period indicates that this type of motion consists of an extremely rapid tremor of the gyro axle. It is of no importance practically and is soon wiped out by friction.

52. Simpler forms of the differential equations and their solution

The magnitudes of some of the terms in the above numerical example are seen to be negligible in comparison with others. The coefficient of $d^4\alpha/dt^4$, for example, is very small in comparison with that of $d^2\alpha/dt^2$. Furthermore, the terms $AC\omega\Omega\cos\lambda$ and Awa are exceedingly small in comparison with $C^2\omega^2$. These facts suggest that the small terms just mentioned may be neglected without appreciable error. On neglecting these terms in (50·1) and (50·2), and then dividing the resulting equations throughout by $C\omega$, we get

$$C\omega\frac{d^2\alpha}{dt^2}+\Omega wa\cos\lambda\cdot\alpha=0, \tag{52·1}$$

$$C\omega\frac{d^2\chi}{dt^2}+\Omega wa\cos\lambda\cdot\chi=0. \tag{52·2}$$

Here the auxiliary equation in each case is

$$C\omega r^2+\Omega wa\cos\lambda=0.$$

$$\therefore\quad r=\pm i\sqrt{\frac{\Omega wa\cos\lambda}{C\omega}}.$$

The period is therefore

$$T=2\pi\sqrt{\frac{C\omega}{\Omega wa\cos\lambda}}. \tag{52·3}$$

Now using the numerical data of Art. 51, we have

$$T=2\pi\sqrt{\frac{222.5}{0.00007292\times 6.5\cos 41^\circ}}$$

$$=4955.4 \text{ seconds}$$

$$=82.6 \text{ minutes,}$$

which is the same value as found for the long period in the previous case. The dropping of the negligible quantities in (50·1) thus caused only the loss of the rapid tremor oscillations, which is of no importance. We shall therefore use equations (52·1) and (52·2) instead of (50·1) and (50·2). The period of oscillation both vertically and horizontally is given by (52·3).

We shall now solve for α and ψ with assumed initial conditions. From (52·1) we have

$$C\omega r^2+\Omega wa\cos\lambda=0.$$

Hence
$$r = \pm i\sqrt{\frac{\Omega wa\cos\lambda}{C\omega}} = \pm ik, \quad \text{say.}$$

Then
$$\alpha = C_1\sin kt + C_2\cos kt,$$

and
$$\frac{d\alpha}{dt} = C_1 k\cos kt - C_2 k\sin kt.$$

Now assuming that $\alpha = \alpha_0$ and $d\alpha/dt = 0$ when $t = 0$, we have

$$\alpha_0 = C_2,$$

$$0 = C_1 k.$$

Hence $C_1 = 0$, and

$$\alpha = \alpha_0\cos kt, \tag{52·4}$$

which represents simple harmotion with amplitude α_0.

We wish to find χ without further integration, in order to avoid bringing in additional constants of integration. Consequently we go back to equations (48·6). From (52·4) we have:

$$\frac{d\alpha}{dt} = -\alpha_0 k\sin kt, \quad \frac{d^2\alpha}{dt^2} = -\alpha_0 k^2\cos kt, \quad \frac{d^3\alpha}{dt^3} = \alpha_0 k^3\sin kt.$$

Now differentiating (48·6) (b) with respect to time, we have

$$A\frac{d^3\alpha}{dt^3} - C\omega\frac{d^2\chi}{dt^2} + C\omega\Omega\cos\lambda\frac{d\alpha}{dt} = 0.$$

Eliminating $d^2\chi/dt^2$ between this equation and (48·6) (a), we get

$$A^2\frac{d^3\alpha}{dt^3} + C^2\omega^2(1 + \Omega\cos\lambda)\frac{d\alpha}{dt} + wa\chi = 0.$$

Replacing $d^3\alpha/dt^3$ and $d\alpha/dt$ by their values as found above and solving the result for χ, we have

$$\chi = \frac{\alpha_0 k}{wa}\left(C\omega + A\Omega\cos\lambda - \frac{A^2k^2}{C\omega}\right)\sin kt.$$

When k is replaced by its value $\sqrt{\dfrac{\Omega wa\cos\lambda}{C\omega}}$, the value of χ becomes

$$\chi = \frac{\alpha_0}{wa}\sqrt{\left(\frac{\Omega wa\cos\lambda}{C\omega}\right)}\left(C\omega + A\Omega\cos\lambda - \frac{A^2\Omega wa\cos\lambda}{C^2\omega^2}\right)\sin kt.$$

Neglecting the second and third terms in the parentheses because of their smallness in comparison with $C\omega$ and then reducing the result slightly, we get

$$\chi = \alpha_0 \sqrt{\left(\frac{C\omega\Omega\cos\lambda}{wa}\right)} \sin kt. \tag{52·5}$$

This equation (52·5) shows that $\chi = 0$, or $\beta = \beta_0$, when $t = 0$. Hence when the gyro axle reaches its extreme position to the right, it is then in the plane $\beta = \beta_0$. The above equation (52·5) also shows that the spin axis executes simple harmonic motion in the vertical direction.

Eliminating t by solving (52·4) and (52·5) for $\cos kt$ and $\sin kt$, respectively, and then squaring and adding the resulting equations, we get

$$\frac{\alpha^2}{\alpha_0^2} + \frac{\chi^2}{\alpha_0^2\left(\frac{C\omega\Omega\cos\lambda}{wa}\right)} = 1,$$

which is the equation of an ellipse in a plane perpendicular to the spin axis in its settled position. The north end of the spin axle therefore traces an ellipse as it executes simple harmonic motion in the horizontal and vertical directions.

53. Comparison of the amplitudes of the horizontal and vertical oscillations. Locus of the end of the gyro axle

Since the quantity $\sqrt{\frac{C\omega\Omega\cos\lambda}{wa}}$ is very small in comparison with unity, the amplitude of χ is very much less than α_0, the amplitude of α. For the data of the numerical example worked in Art. 51 we have

$$\sqrt{\frac{C\omega\Omega\cos\lambda}{wa}} = \sqrt{\frac{222.5 \times 0.00007292 \cos 41^\circ}{6.5}}$$

$$= 0.0434 = \tfrac{1}{23}, \quad \text{approximately.}$$

Hence the amplitude of χ is only $\frac{1}{23}$ that of α.

Equations (52·4) and (52·5) show that α and χ are out of phase by a quarter of a period; for when $kt = 0$, $\alpha = \alpha_0$ and $\chi = 0$, whereas when $kt = \frac{1}{2}\pi$, $\alpha = 0$ and χ is a maximum. This means that the north end of the gyro axle describes a very elongated ellipse in a plane perpendicular to the axle, the major axis of the ellipse lying in the plane $\beta = \beta_0$.

To find β_0 for the data of the example in Art. 51, we have, from equation (47·2),

$$\beta_0 = \frac{222.5 \times 0.00007292 \sin 41^\circ}{6.5}$$

$$= 0.0016376 \text{ radian}$$

$$= 5'.6.$$

54. A third method of deriving the differential equations of motion

If $\alpha_0 = 2^\circ$, for example, the maximum value of χ will be

$$\frac{120'}{23} = 5'.2.$$

Since the period of oscillation of gyrocompasses is about 84 minutes, the time required for the axle to move through $5'.2$ will be 21 minutes. The *average* angular velocity of the upward motion for a quarter period is thus

$$\frac{5'.2}{21 \text{ minutes}} = \frac{0.00151 \text{ radian}}{1260 \text{ seconds}} = 0.0000012 \text{ radian per second.}$$

Since this angular velocity is so very small, the upward angular acceleration $d^2\chi/dt^2$ is also exceedingly small and is therefore negligible in equation (48·6) (a) in comparison with the other terms in that equation. This fact gives us a short method of finding the final differential equation of a disturbed motion and the period of oscillation.

On putting $d^2\chi/dt^2 = 0$ in (48·6) (a), we get

$$C\omega \frac{d\alpha}{dt} + wa\chi = 0. \tag{54·1}$$

Hence by differentiation,

$$C\omega \frac{d^2\alpha}{dt^2} + wa\frac{d\chi}{dt} = 0,$$

or

$$\frac{d\chi}{dt} = -\frac{C\omega}{wa}\frac{d^2\alpha}{dt^2}.$$

Substituting this value of $d\chi/dt$ into (48·6) (b), we get

$$\left(A + \frac{C^2\omega^2}{wa}\right)\frac{d^2\alpha}{dt^2} + C\omega\Omega\cos\lambda \cdot \alpha = 0.$$

Now A is totally negligible in comparison with $C^2\omega^2/wa$. Hence we neglect the A in the parentheses and then divide the resulting equation throughout by $C\omega$, thereby obtaining

$$C\omega\frac{d^2\alpha}{dt^2}+\Omega wa\cos\lambda\cdot\alpha=0, \tag{54·2}$$

which is (52·1). After solving this equation for α, we could find $d\alpha/dt$ and then substitute it into (54·1) to find χ.

55. Motion of the gyroscopic compass with damping

In the preceding pages of this chapter we have considered the motion of an undamped gyrocompass. Such a compass, when once disturbed, would continue to oscillate with undiminished amplitude if not slowed down by frictional forces. For practical reasons it is important that the oscillations be damped out as quickly as possible. Several methods of damping have been devised and used. The simplest and one of the best is the method devised by Elmer A. Sperry, which we shall now explain.

In Arts. 47 and 48 we assumed that the pendulous weight w acted in the XZ-plane. In order to produce the precession necessary to keep the spin axis in the meridian and at the same time damp out the oscillations that might occur, Sperry devised an eccentrically weighted bail (semicircular ring) and fastened it rigidly to the inner ring of the suspension as indicated in Fig. 47. The damping was accomplished by placing the weight w on the bail a short distance to the east of the XZ-plane. In that position the weight will exert a small moment about OZ, the effect of which will be a slight downward precession of the spin axis. This precession will tend to decrease the angle of inclination β and at the same time automatically decrease the azimuth angle α, because of the fact that the ratio of the semiaxes of the ellipse described by the north end of the spin axis is constant, this ratio being $\sqrt{(C\omega\Omega\cos\lambda/wa)}$ (Art. 52). The path of the north end of the spin axis with damping is therefore no longer an ellipse but is a spiral whose radius vector continually decreases to zero.

To find the equations of motion of the spin axis under damping, let δ denote the small angle (about 1°) which the off-center arc of the bail subtends at O (Fig. 47). Then the distance of the weight w from the z-axis is practically $a\delta$. If w be resolved into rectangular components $w\cos\beta$ and $w\sin\beta$, the latter is directed backward

toward the south and thus has a clockwise moment about OZ as viewed from above and is therefore positive. The moment of $w\cos\beta$ about either OY or OZ is zero, but the moments of $w\sin\beta$ about these axes are

$$M_y = -w\sin\beta \times a\cos\delta = -wa\beta, \qquad \text{practically,}$$
$$M_z = w\sin\beta \times a\delta \qquad = wa\beta\delta, \qquad \text{practically.}$$

On substituting these values into the right-hand members of equations (48·3), which came from (16·4), we get

$$\left.\begin{aligned} A\frac{d^2\beta}{dt^2} + C\omega\frac{d\alpha}{dt} - C\omega\Omega\sin(\lambda-\beta) &= -wa\beta, \\ A\frac{d^2\alpha}{dt^2} - C\omega\frac{d\beta}{dt} + C\omega\Omega\sin\alpha\cos(\lambda-\beta) &= wa\beta\delta. \end{aligned}\right\} \qquad (55{\cdot}1)$$

Now expanding $\sin(\lambda-\beta)$ and $\cos(\lambda-\beta)$, neglecting the terms containing the product $\Omega\sin\beta$, putting $\cos\beta = 1$, and $\sin\beta = \beta$, we get

$$\left.\begin{aligned} A\frac{d^2\beta}{dt^2} + C\omega\frac{d\alpha}{dt} - C\omega\Omega\sin\lambda + wa\beta &= 0, \\ A\frac{d^2\alpha}{dt^2} - C\omega\frac{d\beta}{dt} + C\omega\Omega\sin\alpha\cos\lambda - wa\beta\delta &= 0. \end{aligned}\right\} \qquad (55{\cdot}2)$$

To find the equilibrium position of the spin axis we put

$$\frac{d^2\alpha}{dt^2} = 0, \quad \frac{d\alpha}{dt} = 0, \quad \frac{d^2\beta}{dt^2} = 0, \quad \frac{d\beta}{dt} = 0$$

in (55·2), thus obtaining the equations

$$C\omega\Omega\sin\lambda = wa\beta_0,$$
$$C\omega\Omega\sin\alpha_0\cos\lambda = wa\beta_0\delta.$$

From the first of these we get

$$\beta_0 = \frac{C\omega\Omega\sin\lambda}{wa}. \qquad (55{\cdot}3)$$

Substituting this value into the second equation we get

$$\sin\alpha_0 = \delta\tan\lambda.$$

But since δ is very small, $\sin\alpha_0$ is also small and may be replaced by α_0. Then we have

$$\alpha_0 = \delta\tan\lambda. \qquad (55{\cdot}4)$$

The spin axis therefore points slightly to the east of the meridian when it comes to rest, but this small deviation is easily corrected.

If we put $\lambda = 0$ in (55·3) and (55·4), we get $\alpha_0 = 0$ and $\beta_0 = 0$, which shows that at the equator the spin axis is horizontal and points true north.

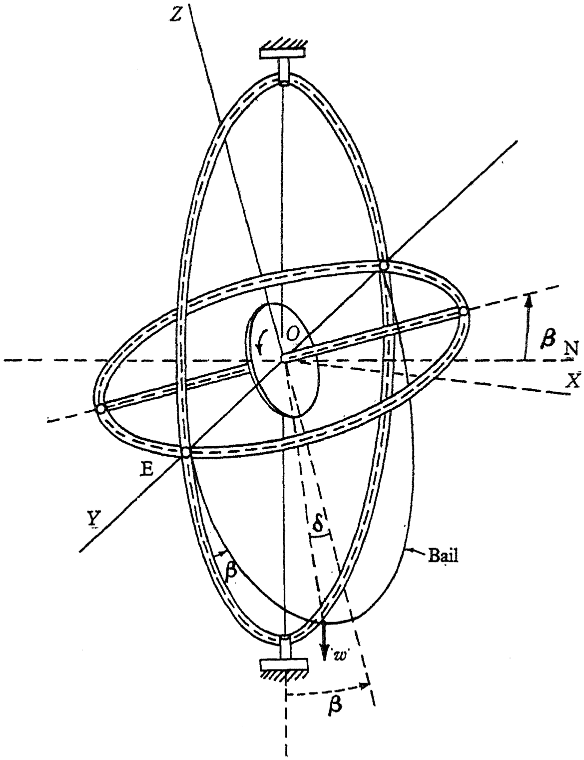


Fig. 47

Since β changes very slowly, $d^2\beta/dt^2$ is negligible in comparison with the other terms in the first of equations (55·2). Also, since α is a small angle, we may replace $\sin\alpha$ by α in the second equation. Then equations (55·2) become

$$\left.\begin{aligned} &C\omega\frac{d\alpha}{dt} - C\omega\Omega\sin\lambda + wa\beta = 0, \\ &A\frac{d^2\alpha}{dt^2} - C\omega\frac{d\beta}{dt} + C\omega\Omega\alpha\cos\lambda - wa\beta\delta = 0. \end{aligned}\right\} \qquad (55\cdot5)$$

From the first of these we get

$$\beta = \frac{C\omega}{wa}\left(-\frac{d\alpha}{dt} + \Omega \sin \lambda\right).$$

Then
$$\frac{d\beta}{dt} = -\frac{C\omega}{wa}\frac{d^2\alpha}{dt^2}.$$

Substituting these into the second equation, we get

$$\left(A + \frac{C^2\omega^2}{wa}\right)\frac{d^2\alpha}{dt^2} + C\omega\delta\frac{d\alpha}{dt} + C\omega\Omega\alpha\cos\lambda = C\omega\Omega\delta\sin\lambda.$$

Now since A is negligible in comparison with $C^2\omega^2/wa$, we neglect it in the above equation and then divide throughout by $C\omega$, thereby obtaining

$$\frac{d^2\alpha}{dt^2} + \frac{wa\delta}{C\omega}\frac{d\alpha}{dt} + \frac{wa\Omega\cos\lambda}{C\omega}\alpha = \frac{wa\Omega\delta\sin\lambda}{C\omega}. \qquad (55{\cdot}6)$$

Solving this equation, we have

$$r^2 + \frac{wa\delta}{C\omega}r + \frac{wa\Omega\cos\lambda}{C\omega} = 0,$$

whence
$$r = -\frac{wa\delta}{2C\omega} \pm \sqrt{\left\{\left(\frac{wa\delta}{2C\omega}\right)^2 - \frac{wa\Omega\cos\lambda}{C\omega}\right\}}.$$

If the spin axis is to oscillate, the radicand must be negative. Hence we have

$$r = -\frac{wa\delta}{2C\omega} \pm i\sqrt{\left\{\frac{wa\Omega\cos\lambda}{C\omega} - \left(\frac{wa\delta}{2C\omega}\right)^2\right\}},$$

and therefore the complementary function is

$$\alpha_c = \exp\left[-\frac{wa\delta}{2C\omega}t\right](C_1\sin kt + C_2\cos kt),$$

where
$$k = \sqrt{\left\{\frac{wa\Omega\cos\lambda}{C\omega} - \left(\frac{wa\delta}{2C\omega}\right)^2\right\}}.$$

Since the right-hand member of (55·6) is constant, we can find a particular solution by writing

$$\alpha = N, \quad \text{a constant}.$$

Substituting this and its derivatives into (55·6), we get

$$N = \delta\tan\lambda.$$

The complete solution of (55·6) is therefore

$$\alpha = \exp\left[-\frac{wa\delta}{2C\omega}t\right](C_1 \sin kt + C_2 \cos kt) + \delta \tan \lambda. \qquad (55\cdot7)$$

The period of a complete oscillation is

$$T = \frac{2\pi}{k} = \frac{2\pi}{\sqrt{\left\{\dfrac{wa\Omega\cos\lambda}{C\omega} - \left(\dfrac{wa\delta}{2C\omega}\right)^2\right\}}}$$

and increases with δ. When

$$\frac{wa\Omega\cos\lambda}{C\omega} = \left(\frac{wa\delta}{2C\omega}\right)^2$$

or

$$\delta = 2\sqrt{\frac{C\omega\Omega\cos\lambda}{wa}},$$

the spin axis becomes aperiodic or dead beat. A value of δ as small as 7° is sufficient to make the Sperry gyrocompass dead beat.

The period of a damped gyrocompass is about 90 minutes.

To find the damping factor of the oscillations, let T denote the period of a complete oscillation. Then if we count time from the instant $t = t_0$, the time at the end of the first period is $t = t_0 + T$. Hence the amplitude of the oscillation at the end of the first period is, by (55·7),

$$\alpha = \exp\left[-\frac{wa\delta}{2C\omega}(t_0 + T)\right][C_1 \sin k(t_0 + T) + C_2 \cos k(t_0 + T)] + \delta \tan \lambda$$

$$= \exp\left[-\frac{wa\delta}{2C\omega}T\right] \exp\left[-\frac{wa\delta}{2C\omega}t_0\right](C_1 \sin kt_0 + C_2 \cos kt_0) + \delta \tan \lambda.$$

The damping factor for a complete oscillation is therefore

$$\exp\left[-\frac{wa\delta}{2C\omega}T\right],$$

and for a half period (the time of swing from extreme right to extreme left) it is $\exp\left[-\dfrac{wa\delta}{4C\omega}T\right]$. This latter is called the damping factor and is denoted by f. We thus have

$$f = \exp\left[-\frac{wa\delta}{4C\omega}T\right].$$

The value of f is usually about $\frac{1}{3}$. About 3 hours is required for the disturbed spin axis to settle to its equilibrium position.

56. The speed and course error

In the preceding discussion of the gyroscopic compass we have assumed that it was installed ashore on a stationary base. A compass installed on a ship is subject to certain errors due to the motion of the ship. For example, when a ship is sailing due north at a constant speed of v feet per second, it has an angular velocity of v/R radians per second about an axis through the center of the earth and perpendicular to the meridian plane, R being the radius of the earth in feet. This angular velocity is represented by a vector directed to the west and therefore along the negative direction of the y-axis (Fig. 46). It could therefore be inserted as a negative term in ω_y in equations (48·1) and its deviating effect found from the solution of the amended equations. The deviation of the gyro axis can be found more simply, however, by the following method:

In latitude λ the horizontal projection of the vector representing the earth's angular velocity is $\Omega \cos \lambda$, directed true north. The gyro axis points in this direction when the ship is at rest. When the ship is sailing north with speed v, however, the gyro axis points in the direction of the resultant of the combined vectors v/R and $\Omega \cos \lambda$, as seen from the vector triangle Fig. 48. The westward deviation of the gyro axis from the meridian is therefore given by the equation

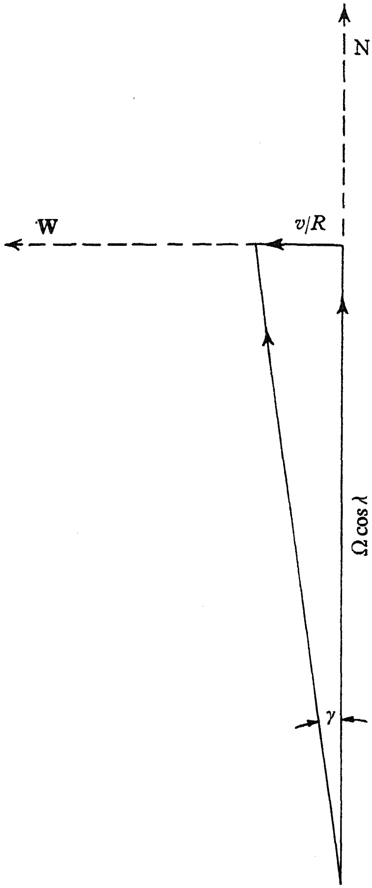


Fig. 48

$$\tan \gamma = \frac{v/R}{\Omega \cos \lambda} = \frac{v}{R\Omega \cos \lambda}.$$

If the course of the ship is not true north but makes an angle θ with the meridian, the northward component of the speed is $v\cos\theta$. Hence in this case the deviation angle is given by the equation

$$\tan\gamma = \frac{v\cos\theta}{R\Omega\cos\lambda}.$$

Since γ is a small angle not exceeding 2° or 3°, we may replace $\tan\gamma$ by γ and thus get

$$\gamma = \frac{v\cos\theta}{R\Omega\cos\lambda}.$$

If the course of the ship is south, or in a southerly direction, the compass deviation will be to the east of the meridian, as can be seen from a vectorial triangle.

The speed and course error is easily corrected by mechanical means. The Sperry gyrocompass is equipped with a corrector which, when set by hand for the ship's speed and latitude, makes the required correction automatically, thereby causing the compass to indicate the true course of the ship.

57. The Sperry mercury ballistic gyroscopic compass

The bail type Sperry gyrocompass is subject to error on a rolling ship when the ship is proceeding in an intercardinal direction. The error is caused by accelerations produced by the rolling. To prevent this type of error the weighted bail has been replaced in the more recent Sperry compasses by a pair of U-tubes with cylindrical cups at the ends and partially filled with mercury, as indicated in exaggerated size in Fig. 49. The planes of the U's are parallel to the spin axis of the gyro, and the U-tubes are rigidly fastened to the gyro frame so as to tilt with the gyro axle.

It is plain from Fig. 45 that when a free gyro is spinning at high speed the north end of its axle will rise as the earth turns eastward. This upward tilting of the axle and the U-tubes with it causes mercury to flow southward, increasing the weight of the south cups at the expense of the north cups. This unequal weight of the cups produces a moment about the axis of nodes and thus causes precession of the gyro spin axis. If the rotation of the gyro is counterclockwise or westward, the precession of the spin axis will also be westward. When the size of the cups and the quantity of mercury are so adjusted that the westward precession is just sufficient to

keep the gyro axis in the meridian, the compass will indicate true north. The reader will note that whereas the gyro rotates to the eastward in the bail type of compass it must rotate to the westward in the mercury type.

The inertia and viscosity of the mercury prevent the flow from being instantaneous from the north cups to the south cups, and this

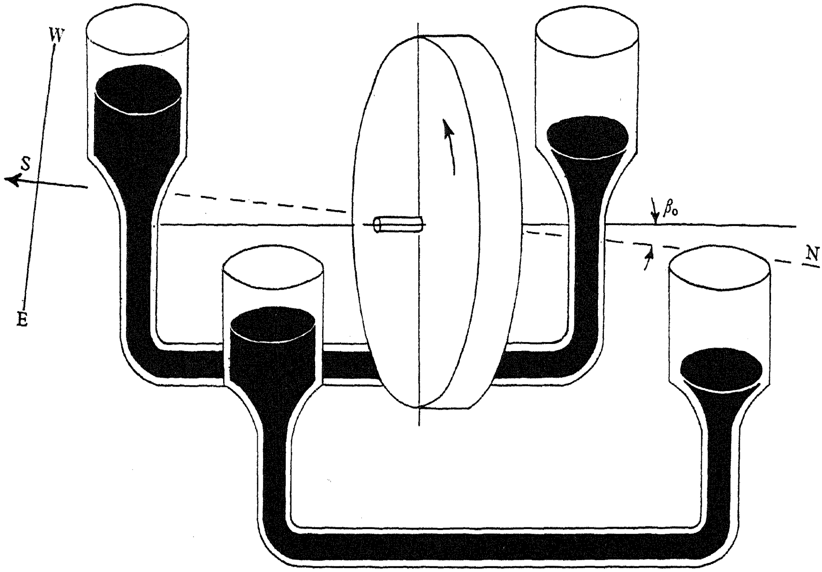


Fig. 49

means that there is a time lag between the tilting and the moment it produces. This time lag and the decreased amplitude of the mercury surging are the factors that prevent intercardinal rolling error. The mercury ballistic gyrocompass is comparatively insensitive to quick alternating accelerations such as are produced by a rolling ship.

The damping of this type of gyrocompass is accomplished by an eccentric arrangement that produces a small moment about the z-axis, just as in the bail type.

The mathematical theory of the mercury ballistic Sperry compass is very similar to that of the bail type and leads to the same results. The interested reader will find this compass fully described and treated in pp. 88–100 of the *The Theory of the Gyroscopic Compass* (1944), by A. L. Rawlings, one of the inventors of the mercury ballistic type.

B. GYROSCOPIC STEERING

58. Rate gyroscopes

A rate gyroscope is a gyroscope constrained to one degree of freedom. Although such a gyroscope has only one degree of complete freedom (the freedom to spin about its axis of rotation),

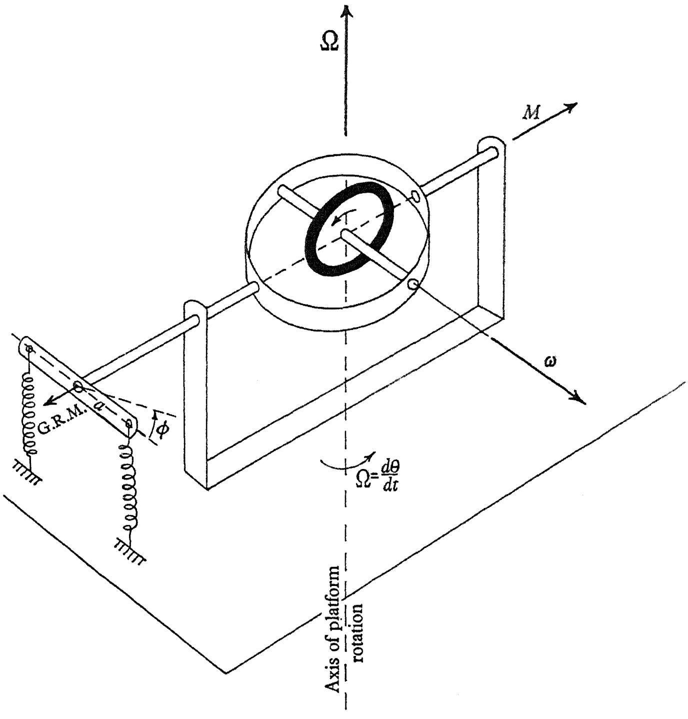


Fig. 50

a restrained and limited rotation is permitted about its torque axis, or axis of nodes. Such rotation is limited by springs which can exert a moment about the torque axis.

Fig. 50 is a schematic representation of the mounting of a rate gyroscope. The outer gimbal frame is rigidly fastened to a platform, which may be the deck of a ship or the floor of an airplane. If an airplane, for example, is flying horizontally and makes a turn to the right or left, the platform turns with it and thus rotates about a vertical axis through the center of gravity of the gyro. Such a

rotation is a forced precession of the spin axis and therefore induces a gyroscopic reaction moment about the torque axis. The rotation caused by this induced G.R.M. is resisted by the springs shown in the figure.

Let: k = spring constant,

a = distance from torque axis to point of attachment of spring,

F = force exerted by spring,

ϕ = tilt angle.

Then $a\phi$ = stretch of spring (approximately),

and $F = ka\phi$, by Hooke's law.

The moment of F about the torque axis is therefore

$$M = a \times F = ka^2\phi = b\phi, \quad \text{say.}$$

But this moment is equal to the G.R.M. $= C\omega\dot{\psi} = C\omega\,(d\psi/dt)$, where ψ denotes the precession angle. We therefore have

$$b\phi = C\omega \frac{d\psi}{dt}. \tag{58·1}$$

This equation shows that the tilt angle is directly proportional to the rate of turn of the platform. When the tilt angle is shown on a dial, the aviator can therefore see at a glance the magnitude and direction of the turn his plane is making.

Since the rotation of the platform will usually be at a variable rate, we find from (58·1)

$$\frac{d\phi}{dt} = \frac{C\omega}{b} \frac{d^2\psi}{dt^2}. \tag{58·2}$$

The rate of change of the tilt angle therefore indicates the acceleration of the platform rotation.

Rate gyroscopes have many aeronautical and military applications. They are the controlling elements in the following devices: the Turn Indicator used in blind flying, the Rate Gyro Gunsight that automatically computes the 'lead' required for hitting a moving target, the Gun-Bomb-Rocket sights used on jet fighter planes, the control gyro for the Sperry Ship Stabilizer, the Yaw Damper for stabilizing yawing oscillations in jet aircraft with swept-back wings, and the Hermetic Integrating Gyro (HIG) used in platform stabilization and in rate measurements.

59. Motion of a damped rate gyroscope

Imperfections in a rate gyroscope may cause tremulous vibration or quiver of the torque or output axis. To reduce such disturbance as much as possible, rate gyroscopes are equipped with a damping mechanism (dashpot or otherwise). To find the motion of a damped rate gyroscope, let

c = damping coefficient,

I = moment of inertia of gimbal system (gimbal, rotor and axle, and crossbar as shown in Fig. 47) about torque axis.

Then the G.R.M. $C\omega\dot{\psi}$ induced by forced precession must maintain the angle ϕ, produce an angular acceleration of the gimbal assembly about the torque axis, and overcome the damping moment about that axis. Hence by equation (20.2) the differential equation of the oscillations about the torque axis is

$$I\frac{d^2\phi}{dt^2}+c\frac{d\phi}{dt}+b\phi=C\omega\dot{\psi},$$

or

$$\frac{d^2\phi}{dt^2}+\frac{c}{I}\frac{d\phi}{dt}+\frac{b}{I}\phi=\frac{C\omega}{I}\dot{\psi}. \tag{59·1}$$

Solving this equation by the usual procedure, we have

$$r^2+\frac{c}{I}r+\frac{b}{I}=0,$$

whence

$$r=\frac{-\frac{c}{I}\pm\sqrt{\left\{\left(\frac{c}{I}\right)^2-\frac{4b}{I}\right\}}}{2}=-\frac{c}{2I}\pm i\sqrt{\left\{\frac{b}{I}-\left(\frac{c}{2I}\right)^2\right\}}.$$

Hence the complementary part of the solution is

$$\phi_c=e^{-ct/2I}\,(C_1\cos mt+C_2\sin mt),$$

where

$$m=\sqrt{\left\{\frac{b}{I}-\left(\frac{c}{2I}\right)^2\right\}}.$$

The precessional velocity $\dot{\psi}$ may here be treated as a constant, since a small disturbance about the torque axis could have little effect on the precession of the base to which the gyroscope is attached. Hence to find a particular solution of (59·1) we put

$$\phi=A,\quad \text{a constant.}$$

Then

$$\frac{d\phi}{dt}=\frac{d^2\phi}{dt^2}=0.$$

Substituting these quantities into (59·1), we find

$$\frac{bA}{I}=\frac{C\omega\dot{\psi}}{I},$$

whence

$$A=\frac{C\omega}{b}\dot{\psi}.$$

The complete solution of (59·1) is therefore

$$\phi=\frac{C\omega\dot{\psi}}{b}+e^{-ct/2I}(C_1\cos mt+C_2\sin mt). \tag{59·2}$$

The disturbance thus soon dies out, leaving the necessary relation

$$\phi=\frac{C\omega\dot{\psi}}{b}.$$

The period of the oscillation is

$$P=\frac{2\pi}{m}=\frac{2\pi}{\sqrt{\left\{\frac{b}{I}-\left(\frac{c}{2I}\right)^2\right\}}}=2\pi\sqrt{\frac{I}{b}} \tag{59·3}$$

practically, since $(c/2I)^2$ is negligible in comparison with b/I.

To find the damping factor we note that the damping coefficient after a complete period is

$$\exp\left[-\frac{c}{2I}(t+P)\right]=\exp\left[-\frac{ct}{2I}\right]\times\exp\left[-\frac{cP}{2I}\right].$$

Hence the damping factor for a complete period is $e^{-cP/2I}$ and for a half-period (from one extreme of the swing to the other) is $e^{-cP/4I}$. When P is replaced by its value from (59·3), the exponent $cP/4I$ becomes

$$\frac{c}{4I}\times 2\pi\sqrt{\frac{I}{b}}=\frac{c\pi}{2\sqrt{(bI)}},$$

and we then have for the damping factor

$$f=\exp\left[-\frac{c\pi}{2\sqrt{(bI)}}\right].$$

60. The rate gyroscope as a differentiator and integrator

The gyroscope, as is the case with all machines, is an instrument which receives an input and delivers a resulting output. If the input is a moment or torque, the output is a precession; whereas if the input is a forced precession, the output is a reaction moment.

The equation
$$b\phi = C\omega \frac{d\psi}{dt}$$
of Art. 58 shows that if the input of a rate gyroscope is an angle ϕ (due to an applied moment), the resulting output is an angular velocity $d\psi/dt$ which is proportional to the input angle; and that if the input is an angular velocity $d\psi/dt$ (due to a forced precession), the output is an angle ϕ (produced by the gyroscopic reaction moment) which is proportional to the input angular velocity. These operations and results correspond to the differentiation of an angle in the first case and to the integration of an angular velocity in the second case. The rate gyroscope can thus be used to compute angular velocities from given angles and to compute angles from given angular velocities.

These properties of the rate gyroscope are utilized in automatic gun sights and other computing devices where angles and angular velocities are involved. The constants in the equation can be adjusted to obtain the desired angle or angular velocity.

The equation
$$b\frac{d\phi}{dt} = C\omega \frac{d^2\psi}{dt^2}$$
shows in like manner that the rate gyroscope can compute an angular acceleration from a given angular velocity and compute an angular velocity from a given angular acceleration.

61. Steering of torpedoes

One of the earlier applications of the gyroscope was to the steering of torpedoes. A torpedo is equipped with two pairs of rudders, a horizontal pair for controlling the depth of the torpedo as it moves through the water to its target and a vertical pair for keeping the torpedo on its course to the target. Both pairs are controlled by a single gyroscope.

The gyroscope carried by a torpedo is mounted in gimbals (Cardan suspension) with its spin axis parallel to the longitudinal axis of the torpedo, the outer ring of the gimbals being free to rotate about a vertical axis in a frame rigidly attached to the torpedo. The gyro is small, only 3 or 4 inches in diameter, but it spins at 14,000–20,000 r.p.m. At the instant when the torpedo is fired, or a few seconds before, the gyro is started spinning and reaches its required speed within a few seconds. If the torpedo encounters

any force which deflects it horizontally, the spin axis of the gyro maintains its direction unchanged, but the rotation of the torpedo about the vertical axis of the gyro frame actuates the vertical rudders and causes them to deflect the torpedo in the opposite direction. As soon as the torpedo passes the direction of the spin axis of the gyro in this opposite swing, the rudder action is reversed and deflects the torpedo back in its original direction of swing. This back-and-forth deflection continues throughout the run of the torpedo, with the result that the torpedo travels in a serpentine path to its target.

The depth of the torpedo is controlled in a similar manner by horizontal rudders which are actuated when the torpedo noses upward or downward in its run.

62. Other steering devices

The *Gyro-Pilot* is a gyroscopic device for steering ships automatically. It is used in connection with the gyroscopic compass and is controlled by the compass.

The *Directional Gyro* is used in connection with the magnetic compass in blind flying. It is an azimuth instrument and gives a stable indication of direction.

The *Artificial Horizon*, though not a steering device, provides an artificial horizon to enable the aviator to keep his airplane in level flight when the natural horizon cannot be seen. It is controlled by a gyro having its spin axis vertical, the axis remaining vertical however the plane may be tilted.

The *Automatic Pilot* is a combination of the Directional Gyro and the Artificial Horizon. It employs two gyros, one installed with spin axis horizontal and the other with spin axis vertical. The former maintains the direction of the airplane in azimuth by controlling vertical rudders and the latter keeps the plain horizontal by controlling horizontal rudders. The automatic pilot can steer an airplane for a considerable time without attention from the human pilot.

Another type of automatic pilot (under development or recently completed) for airplanes employs three rate gyroscopes, one for each of the principal axes of the plane. The outer gimbal of each is rigidly fixed to the plane, so that angular motion (forced precession) about any principal axis will be clearly indicated.

Gyroscopic gun sights enable antiaircraft guns to hit flying targets with ease and accuracy. They calculate automatically and instantaneously the proper aiming of the gun in order to hit the target.

The controlling element in the gun sight is a rate gyroscope which computes the 'lead' of the gun required for hitting the moving target.* In aiming the gun the spin axis of the gyro is aligned parallel to the line of sight from gun to target. Then as the line of sight is rotated at an angular velocity $d\psi/dt$ to keep up with the moving taget, the rate gyro computes the lead angle ϕ in accordance with equation (58·1). This application of the gyroscope to gun aiming is due to Dr C. S. Draper of the Massachusetts Institute of Technology.

In the first application of a gun sight of this type (the Sperry Mark 14) in World War II, 32 attacking airplanes were shot down in a single engagement by the antiaircraft guns on one battleship.

An application of the gyroscope to the automatic firing of naval guns on board ship is briefly discussed at the end of Art. 72.

C. SOME RECENT TYPES OF GYROSCOPES

In recent years gyroscopes have undergone many improvements and refinements, and their applications have been widely extended. Refined methods of suspending the rotors and the wide use of electric, magnetic and electronic accessories have increased the accuracy and applicability of modern gyroscopes. Much research has been done on inertial systems of navigation, systems based entirely on the inertial property of a spinning gyro—the property that keeps the spin axis pointing in the same direction with respect to fixed space. The inertial system is self-contained and does not depend on any external object or source of energy. Some of the more recent types of gyroscopes are described briefly below. Doubtless many other advances in the gyroscopic field have not yet been made public.

* In order to hit a moving target with a projectile, the gun must be aimed at a point ahead of the target, because the target will move a certain distance ahead while the projectile is in flight. The distance from the target to the point aimed at is called the lead. The amount of required lead depends on the distance from gun to target and on the speed and direction of flight of the target.

63. The Sperry Mark 22 gyrocompass

This is the smallest gyroscopic compass yet developed. It is simple and rugged in construction, occupies less than half a cubic foot of space, and weighs 9 pounds. It has been designed for use on small vessels of all types—speed boats, landing craft, amphibious vehicles, etc.—and will indicate true north when a speeding boat is making fast and sharp turns. In performance, the Mark 22 compares favorably with the large standard gyroscopic compasses used on large ships.

64. The Arma miniature gyrocompass

This is a small gyrocompass designed especially for military vehicles, such as tanks and P.T. boats. It indicates true north with an accuracy of about half a degree under all vehicular speeds and conditions, and therefore provides accurate bearings and courses for land and water vehicles in uncharted areas.

The rotor spins at 12,000 r.p.m. inside a helium-filled and hermetically sealed ball made of stainless steel. The ball floats in neutral equilibrium in a fluid inside a fluid tank, and is held in position by very fine wires.

The Arma Miniature Gyrocompass is designed to operate at temperatures from $-65°$ to $+130°$ F. and to give reliable indications in all latitudes up to 75°. Important features of this compass are its ruggedness, small size, light weight (67 pounds) and reasonably high accuracy.

The Arma Corporation also has under development a very small and precise gyroscope of a similar type for use in an inertial system of navigation for airplanes and guided missiles.

65. The Draper hermetic integrating gyro (HIG)

This is a very small one-degree-of-freedom gyroscope of high sensitivity, high accuracy and high resistance to shock and vibration. It is used mainly for platform stabilization and rate measurement applications.

The original research leading to the design of the HIG unit was done under the direction of Dr. C. S. Draper at the Massachusetts Institute of Technology. The development was continued by the Minneapolis-Honeywell Regulator Company, by whom it is manufactured.

The chief components of the HIG are the gyro wheel, a torque

generator, a signal generator, and a viscous damper. The gyro wheel is mounted in a sealed container which floats in a viscous fluid of the same average density as the gimbal. The gimbal rotates on jeweled bearings. A magnetic pickup is mounted at one end of the container and a magnetic torque generator is mounted at the other end. The fluid is kept at constant viscosity by temperature control. This fluid opposes the turning of the gimbal about its axis, serves to float the gimbal and thus reduce the pressure on the gimbal bearings, and provides a cushion against shock. The instrument is so designed that the damping torque is equal to the precessional torque.

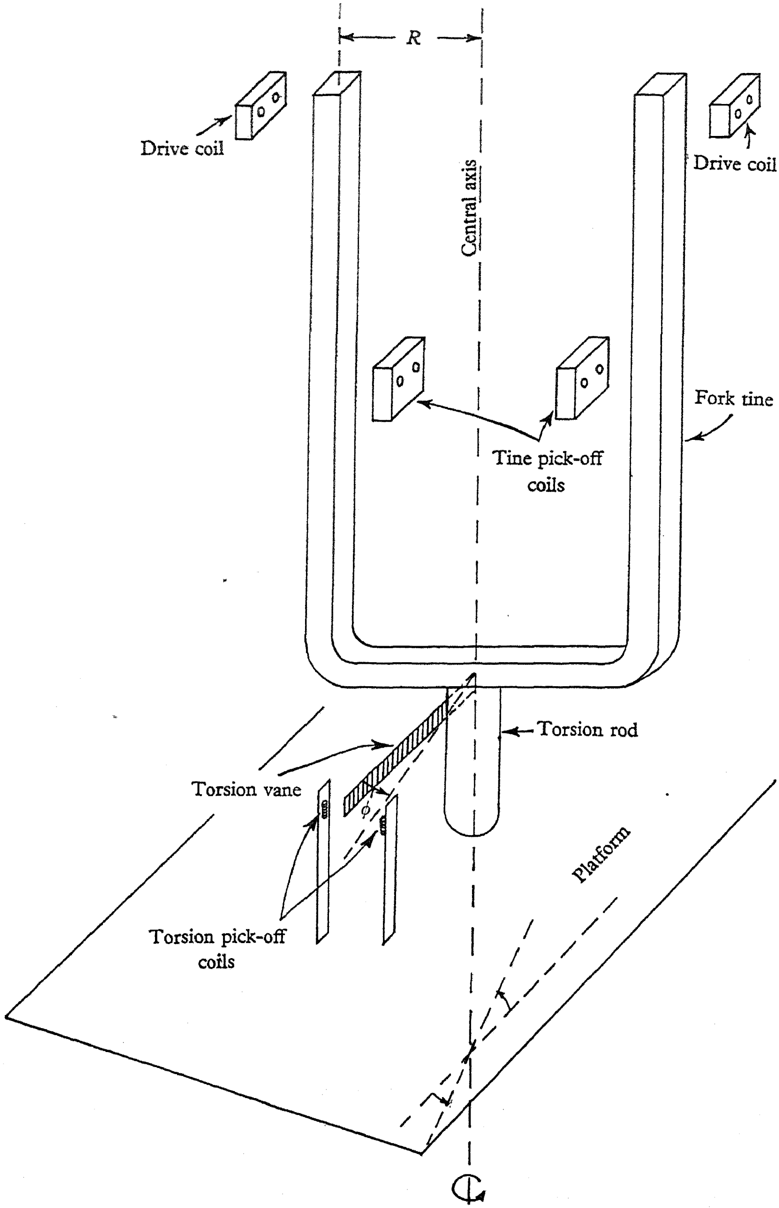
The smallest of the HIG units weighs only $1\frac{1}{4}$ pounds and is compactly housed in a small cylinder slightly more than 2 inches in diameter and $4\frac{1}{4}$ inches long. The rotor spins at 8000 r.p.m.

Gimbal displacement is proportional to the time integral of the input turning rate, and the secondary voltage output of the signal generator is proportional to the angular displacement of the gimbal from a null position. The signal generator serves as the signal source to indicate the angular position of the gimbal with respect to the case. The sensitivity of the HIG is such that a null signal of 1 second of arc can be measured.

66. The Sperry gyrotron vibratory gyroscope

This is an entirely new type of gyroscope and does not even come under the definition of gyroscope given early in this book. Its fundamental element is not a spinning rotor but a kind of tuning fork. It is still in the development stage, but in principle it will probably remain as at present. Its chief application will probably be for measuring the magnitude and direction of the rate of turning, or angular velocity, of the moving body on which it is mounted. For example, when it is mounted vertically on an airplane, it will indicate whether the airplane is turning horizontally about a vertical axis—how fast and in what direction.

Fig. 51 represents schematically the principal parts of the Gyrotron. The tuning fork is attached rigidly to the torsional rod or member, as shown, and the latter is fastened rigidly to the platform. (The platform represents the airplane or other vehicle on which the Gyrotron is mounted.) The torsion rod is designed so as to be in mechanical resonance with the tuning fork; that is, it is designed so that its natural torsional vibration frequency about its central axis is the same as the natural vibration frequency of the fork tines.



R
Drive coil
Drive coil
Central axis
Fork tine
Tine pick-off coils
Torsion rod
Torsion vane
ϕ
Platform
Torsion pick-off coils

Fig. 51

The fork vibrations will then produce torsional vibrations of the same frequency.

The device is provided with three sets of magnetic coils, as indicated in the figure. The tine pickoff coils pick up the natural vibrations of the tines and send them through an amplifier. The amplified signals go to the magnetic drive coils, which keep the tines vibrating at their natural frequency. The third set of coils is the torsion pickoff coils from which the output signal voltage is obtained. The tines must vibrate in the same plane, with the same frequency and amplitude, and in opposite directions; they must move toward and away from each other simultaneously. The tuning fork and its electrical pickoffs are all hermetically sealed in a suitable container.

The operation of the Gyrotron is as follows: When the platform is not turning, the fork tines vibrate in one plane. But when the platform turns, it forces the vibrating fork to rotate at the same rate about its central axis. Then when the tines are approaching each other, the moment of inertia of the fork about the central axis is decreasing. This decrease in the moment of inertia causes the angular velocity of the fork to speed up in obedience to the law of conservation of angular momentum (Art. 15, equation (15·3)); and, conversely, when the tines are moving away from each other, the moment of inertia is increasing and thereby slowing down the angular velocity of the fork about its axis.* This alternate speeding up and slowing down of the angular velocity of the fork about its axis produces relative motion between the plane of the tines and the supporting platform, and this means that the particles composing the tines are periodically accelerated with respect to the

* From the equation $H = I\omega$ (Art. 14, equation (14·6)), we have $\omega = H/I$, and since H is constant in this case, it is obvious that ω varies inversely as I.

The moment of inertia of a particle m of one of the tines with respect to the central axis is evidently given by the relation [see Figs. 51 and 52 and equation (66.1)]

$$\begin{aligned} I &= mr^2 \\ &= m[R + (\Delta R)\sin\omega_1 t]^2 \\ &= mR^2\left[1 + \frac{\Delta R}{R}\sin\omega_1 t\right]^2 \\ &= mR^2\left(1 + 2\frac{\Delta R}{R}\sin\omega_1 t\right), \quad \text{practically,} \end{aligned}$$

since $(\Delta R/R)^2$ is negligible in comparison with 1. The moment of inertia of the tines about the central axis is given by a similar expression.

platform and also with respect to fixed space. The periodic acceleration of the mass particles of the tines produces an alternating twisting moment about the axis of the torsion rod and thus causes the torsion vane to oscillate between the torsion pickoff coils. The vibrations of the vane cause electric signals to be sent out from the torsion pickoff coils.

If the platform reverses its direction of turn (from clockwise to counterclockwise, or vice versa), all forces will be reversed in direction. This will cause the phase of the torsion vane signals to change by 180°. The change in phase is detected by a phase detector, which compares the phase of the vane signals with that of the tine vibrations.

The tuning fork is designed so as to have a high natural rate of vibration (of the order of 2000 oscillations per second), thereby helping to isolate it from extraneous vibrations.

Since loss of energy in the torsion member is very small, the torsional vibrations must be damped in order to obtain a favorable time constant.

To find the angular velocity of the torsion vane with respect to the platform, let us consider the motion of a particle m of one of the fork tines. Because of the extreme rapidity of tine vibration in comparison with the angular velocity of the platform, the path of m will be practically a very compressed sine curve, as shown in a very expanded and magnified form in Fig. 52. Let R denote the distance from the central axis of the fork to the center of each tine when at rest, and let ΔR denote the amplitude of the fork vibrations. Then the distance of m from the central axis at any time t is evidently given by the relation

$$r = R + (\Delta R)\sin\omega_1 t, \tag{66·1}$$

from which

$$\frac{dr}{dt} = (\Delta R)\,\omega_1\cos\omega_1 t, \tag{66·2}$$

where $2\pi/\omega_1$ denotes the vibration period of the fork.

When the platform is turning, the particle m has a compound acceleration whose radial and transverse components are

$$a_r = \frac{d^2r}{dt^2} - r\omega^2, \tag{66·3}$$

$$a_\theta = r\frac{d\omega}{dt} + 2\omega\frac{dr}{dt}, \tag{66·4}$$

where ω denotes the angular velocity of the platform. The inertial reaction force on m perpendicular to r is therefore $-ma_\theta$, and the moment of this force about the central axis is

$$T = -rma_\theta$$
$$= -rm\left(r\frac{d\omega}{dt} + 2\omega\frac{dr}{dt}\right). \quad (66\cdot5)$$

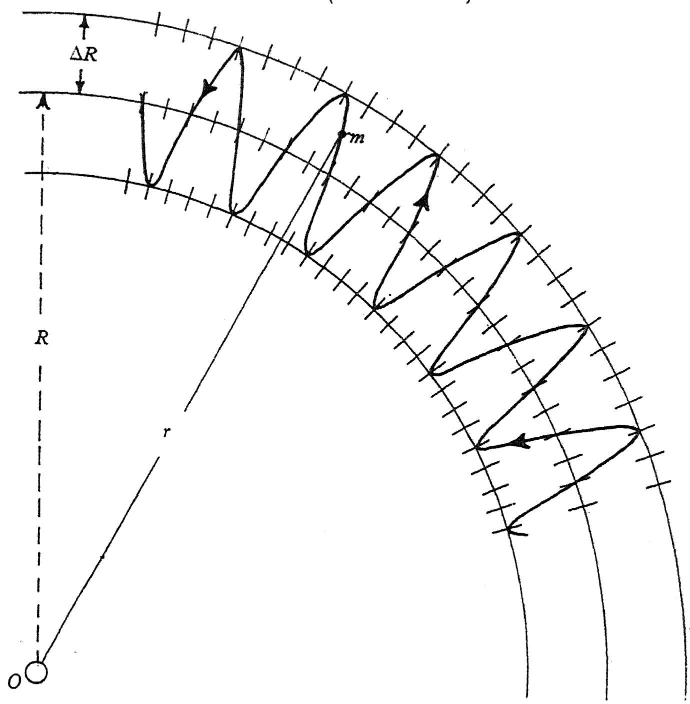


Fig. 52

Although the angular velocity of the platform is in general neither constant nor strictly periodic, it can be represented very approximately by a Fourier series, and for our purpose it is sufficient to use only the fundamental harmonic of the series. Hence we take

$$\omega = \Omega \cos \omega_0 t, \quad (66\cdot6)$$

where Ω denotes the amplitude and $\omega_0/2\pi$ the fundamental frequency of the platform rotation. From this we get

$$\frac{d\omega}{dt} = -\Omega\omega_0 \sin \omega_0 t.$$

Substituting into (66·5) the values of r, dr/dt, ω, $d\omega/dt$, and simplifying slightly, we get

$$T = -mR\Omega\omega_1 \left\{ -R\left(1+\frac{\Delta R}{R}\sin\omega_1 t\right)^2 \frac{\omega_0}{\omega_1}\sin\omega_0 t \right.$$
$$\left. +2(\Delta R)\cos\omega_1 t\cos\omega_0 t + 2(\Delta R)\left(\frac{\Delta R}{R}\right)\sin\omega_1 t\cos\omega_1 t\cos\omega_0 t\right\}. \tag{66·7}$$

Now since $\omega_0 \ll \omega_1$ and $\Delta R \ll R$, the first and third terms in the right member of (66·7) are negligible in comparison with the second term and may therefore be neglected. The torque T thus becomes

$$T = -2mR(\Delta R)\,\Omega\omega_1\cos\omega_1 t\cos\omega_0 t. \tag{66·8}$$

The product of cosines can be transformed to a sum by means of the formula

$$\cos a\cos b = \tfrac{1}{2}\cos(a+b) + \tfrac{1}{2}\cos(a-b).$$

Making this transformation, we get

$$T = -mR(\Delta R)\,\Omega\omega_1[\cos(\omega_1+\omega_0)t + \cos(\omega_1-\omega_0)t]. \tag{66·9}$$

A similar equation holds for all particles of the tines. Hence the reaction moment for the whole fork is

$$T = -MR(\Delta R)\,\Omega\omega_1[\cos(\omega_1+\omega_0)t + \cos(\omega_1-\omega_0)t], \tag{66·10}$$

where M denotes the mass of the fork.

We are now ready to write the differential equation of motion of the torsion vane. Let

I = moment of inertia of fork about central axis,

ϕ = deflection angle of vane at any time t,

c = damping coefficient,

λ = torsion coefficient (torque required to twist the member through 1 radian).

Then since the reaction torque, the damping torque and the moment of the torsional resistance all act to oppose the increase of the deflection angle, we have by equation (20·3)

$$I\ddot{\phi} = -T - c\dot{\phi} - \lambda\phi,$$

or
$$\ddot{\phi} + \frac{c}{I}\dot{\phi} + \frac{\lambda}{I}\phi = \frac{MR(\Delta R)\,\Omega\omega_1}{I}[\cos(\omega_1+\omega_0)t + \cos(\omega_1-\omega_0)t]. \tag{66·11}$$

Since the vibration frequency of the vane would be the same as that of the tines if there were no damping, it is plain* that $\lambda/I=\omega_1^2$. Then if we put $c/I=2k$, $\dfrac{MR(\Delta R)\,\Omega\omega_1}{I}=Q$, $\omega_1+\omega_0=A$ and $\omega_1-\omega_0=B$ for convenience, we can write (66·11) in the simpler form

$$\ddot{\phi}+2k\dot{\phi}+\omega_1^2\phi=Q\,(\cos At+\cos Bt). \tag{66·12}$$

Here the characteristic equation is

$$r^2+2kr+\omega_1^2=0,$$

from which $\quad r=-k\pm i\sqrt{(\omega_1^2-k^2)} \quad \text{if} \quad k<\omega_1.$

Hence the complementary solution is

$$\phi=e^{-kt}[C_1\sin\sqrt{(\omega_1^2-k^2)}\,t+C_2\cos\sqrt{(\omega_1^2-k^2)}\,t]. \tag{66·13}$$

The natural period of oscillation is therefore

$$\tau=\frac{2\pi}{\sqrt{(\omega_1^2-k^2)}}.$$

Since the complementary solution and its derivative will soon fade out, we do not consider it further.

To find the angular velocity of the vane due to the motion of the platform, we find a particular solution of (66·12) by putting

$$\phi=C_3\sin At+C_4\cos At+C_5\sin Bt+C_6\cos Bt. \tag{66·14}$$

Then:

$$\dot{\phi}=C_3A\cos At-C_4A\sin At+C_5B\cos Bt-C_6B\sin Bt,$$
$$\ddot{\phi}=-C_3A^2\sin At-C_4A^2\cos At-C_5B^2\sin Bt-C_6B^2\cos Bt.$$

Substituting these into (66·12) and equating coefficients of like functions of t on the two sides of the equation, we get

$$(\omega_1^2-A^2)\,C_3-2kAC_4=0,$$
$$(\omega_1^2-A^2)\,C_4+2kAC_3=Q,$$
$$(\omega_1^2-B^2)\,C_5-2kBC_6=0,$$
$$(\omega_1^2-B^2)\,C_6+2kBC_5=Q.$$

* If we put $c=0$ in (66·11) and denote the right-hand member by $f(t)$, the equation becomes

$$\ddot{\phi}+\frac{\lambda}{I}\,\phi=f(t).$$

The natural vibration period in this case is $2\pi/\sqrt{(\lambda/I)}$. But the natural vibration period of the fork tines is $2\pi/\omega_1$. Since the two periods must be the same when $c=0$, we must have $\sqrt{(\lambda/I)}=\omega_1$, or $\lambda/I=\omega_1^2$.

Solving these equations for C_3, C_4, C_5, C_6, and substituting their values into (66·14), we find a particular solution to be

$$\phi = \frac{Q}{(\omega_1^2 - A^2)^2 + (2kA)^2}[2kA \sin At + (\omega_1^2 - A^2)\cos At]$$
$$+ \frac{Q}{(\omega_1^2 - B^2)^2 + (2kB)^2}[2kB \sin Bt + (\omega_1^2 - B^2)\cos Bt],$$

which can be written

$$\phi = \frac{Q}{\sqrt{\{(\omega_1^2 - A^2)^2 + (2kA)^2\}}} \sin(At + \alpha)$$
$$+ \frac{Q}{\sqrt{\{(\omega_1^2 - B^2)^2 + (2kB)^2\}}} \sin(Bt + \beta), \quad (66\cdot 15)$$

where $$\alpha = \tan^{-1}\frac{\omega_1^2 - A^2}{2kA}, \quad \beta = \tan^{-1}\frac{\omega_1^2 - B^2}{2kB}. \quad (66\cdot 16)$$

These equations can be still further simplified by replacing A and B by their values and then neglecting all terms in which the negligible fraction ω_0/ω_1 occurs. Thus

$$\sqrt{\{(\omega_1^2 - A^2)^2 + 4k^2A^2\}} = \sqrt{\{[\omega_1^2 - (\omega_1 + \omega_0)^2]^2 + 4k^2(\omega_1 + \omega_0)^2\}}$$
$$= \sqrt{\{4\omega_1^2\omega_0^2 + 4k^2\omega_1^2\}} = 2\omega_1\sqrt{\{k^2 + \omega_0^2\}},$$
$$\sqrt{\{(\omega_1^2 - B^2)^2 + 4k^2B^2\}} = \sqrt{\{[\omega_1^2 - (\omega_1 - \omega_0)^2]^2 + 4k^2(\omega_1 - \omega_0)^2\}}$$
$$= \{4\omega_1^2\omega_0^2 + 4k^2\omega_1^2\} = 2\omega_1\sqrt{\{k^2 + \omega_0^2\}}.$$

Likewise,

$$\frac{\omega_1^2 - A^2}{2kA} = \frac{\omega_1^2 - (\omega_1 + \omega_0)^2}{2k(\omega_1 + \omega_0)} = \frac{-2\omega_0\left(\omega_1 + \frac{\omega_0}{2}\right)}{2k(\omega_1 + \omega_0)}$$
$$= -\frac{\omega_0}{k}\left(1 - \frac{\omega_0}{2(\omega_1 + \omega_0)}\right) = -\frac{\omega_0}{k}.$$

$$\frac{\omega_1^2 - B^2}{2kB} = \frac{\omega_1^2 - (\omega_1 - \omega_0)^2}{2k(\omega_1 - \omega_0)} = \frac{2\omega_0\left(\omega_1 - \frac{\omega_0}{2}\right)}{2k(\omega_1 - \omega_0)} = \frac{\omega_0}{k}\left(1 + \frac{\omega_0}{2(\omega_1 - \omega_0)}\right) = \frac{\omega_0}{k},$$

Hence $$\beta = \tan^{-1}\frac{\omega_0}{k}, \quad \alpha = \tan^{-1}\left(-\frac{\omega_0}{k}\right) = -\beta.$$

On substituting the above simplified expressions into (66·15) and (66·16), we get

$$\phi = \frac{Q}{2\omega_1\sqrt{(k^2+\omega_0^2)}}\{\sin[(\omega_1+\omega_0)t-\beta]+\sin[(\omega_1-\omega_0)t+\beta]\}. \tag{66·17}$$

When the sum of the sine terms in (66·17) is transformed into a product by means of the formula

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2},$$

the equation becomes

$$\phi = \frac{Q}{\omega_1\sqrt{(k^2+\omega_0^2)}}\sin\omega_1 t\cos(\omega_0 t-\beta). \tag{66·18}$$

The angular velocity of the torsion vane with respect to the platform is therefore

$$\begin{aligned}\dot{\phi} &= \frac{Q}{\sqrt{(k^2+\omega_0^2)}}\left[\cos\omega_1 t\cos(\omega_0 t-\beta)-\frac{\omega_0}{\omega_1}\sin\omega_1 t\sin(\omega_0 t-\beta)\right]\\ &= \frac{Q}{\sqrt{(k^2+\omega_0^2)}}\cos\omega_1 t\cos(\omega_0 t-\beta),\end{aligned}$$

since ω_0/ω_1 is negligible.

Finally, when Q and k are replaced by their values, the angular velocity becomes

$$\dot{\phi} = \frac{2MR(\Delta R)\,\Omega\omega_1}{\sqrt{\{c^2+(2I\omega_0)^2\}}}\cos\omega_1 t\cos(\omega_0 t-\beta). \tag{66·19}$$

This equation shows that the amplitude of the angular velocity of the torsion vane with respect to the platform varies directly with the amplitude of the angular velocity of the platform with respect to fixed space.

It is to be noted in (66·18) and (66·19) that ϕ and $\dot{\phi}$ are modulated quantities* of essentially the same form, but that the amplitude of $\dot{\phi}$ is ω_1 times that of ϕ. A torsion pickoff sensitive to velocity may

* A modulated quantity is a combination of two or more oscillating quantities which produces new frequency components not present in the original oscillating quantities. The modulating factor in (66·18) and (66·9) is $\sin\omega_1 t$.

therefore be utilized. This type of pickoff is not subject to the resolution errors of a displacement-sensitive pickoff.

Some of the advantages of the Gyrotron Vibratory Gyroscope are absence of bearings and other wearing parts, small size, ruggedness, and long life expectancy. There is no maintenance problem and the device is sensitive only to rotation about the central axis.

For a fuller description of the Gyrotron and further technical details of its construction and operation, the reader is referred to two papers in the *Aeronautical Engineering Review* for November 1953.

CHAPTER VII

The Gyroscope as a Stabilizer

A. THE GYROSCOPIC SPHERICAL PENDULUM

67. Definitions

A spherical pendulum is a pendulous body suspended from a fixed point and free to swing in any manner about that point, subject only to the condition that its center of gravity remain at a constant distance from the point of suspension. The locus of the c.g. is thus a spherical surface having the fixed point as center. The ordinary pendulum which swings in a vertical plane and the conical pendulum which swings in a horizontal circle are special cases of a spherical pendulum.

A gyroscopic spherical pendulum is a spherical pendulum wherein the pendulous body is a gyroscope and its mountings, so mounted that its axis of spin coincides with the straight line joining the point of suspension and the c.g. of the gyro. The rotation of the earth has a slight effect on the motion of the gyroscopic pendulum, but that effect will be neglected in the following treatment.

68. Differential equations of motion

To derive the differential equations of motion of a gyroscopic spherical pendulum when slightly disturbed from its equilibrium position, we take a set of fixed rectangular axes $O-XYZ$ and a set of moving rectangular axes $O-X'Y'Z'$, the origin O being at the point of suspension of the pendulum. The position of the moving trihedron at any instant can be specified by the two angles ϕ and θ, as indicated in Fig. 53. If the moving trihedron originally coincided with the set of fixed axes, it can be brought to the position shown in the figure by first rotating it about OX through the angle ϕ and then rotating it about OY' through the angle θ. The angles ϕ and θ are assumed to be small,* and the gyro is assumed to be spinning with

* The treatment of the general case, where ϕ and θ may be of any magnitude, requires the use of elliptic functions and will therefore not be given here.

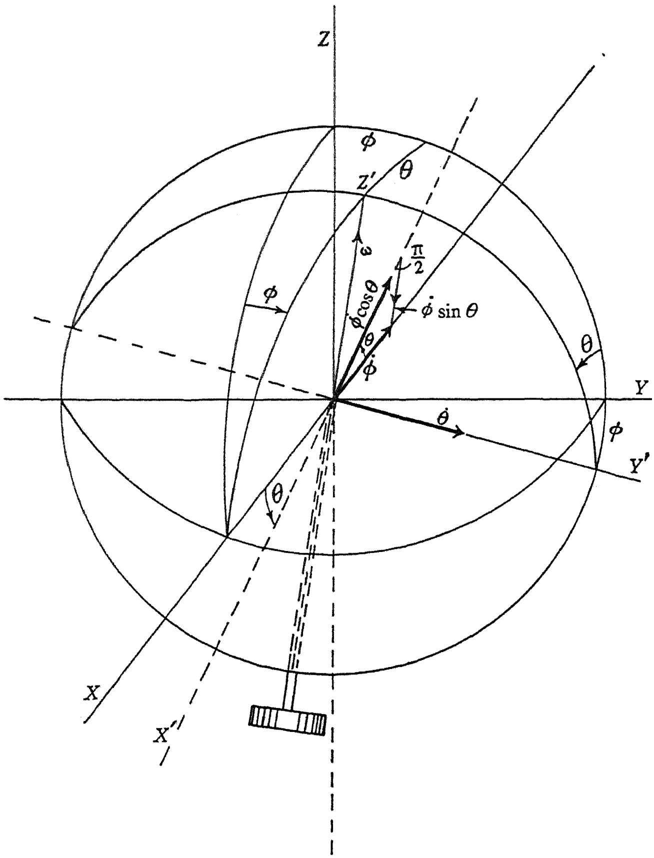


Fig. 53

a constant high velocity ω. The angular velocities $\dot{\phi}$ and $\dot{\theta}$ are also assumed to be small and are indicated by vectors in the figure.

The components of the angular velocity of the moving trihedron along its own axes are seen to be as follows:

$$\omega_{x'} = -\dot{\phi}\cos\theta, \quad \omega_{y'} = \dot{\theta}, \quad \omega_{z'} = -\dot{\phi}\sin\theta.$$

Hence the angular momenta along these axes are

$$H_{x'} = -A\dot{\phi}\cos\theta, \quad H_{y'} = A\dot{\theta}, \quad H_{z'} = C(\omega - \dot{\phi}\sin\theta).$$

Since $\sin\theta$ and $\dot{\phi}$ are both small, their product is negligible in comparison with ω. Hence we may write

$$H_{z'} = C\omega.$$

Then we have

$$\frac{dH_{x'}}{dt} = -A\ddot{\phi}\cos\theta + A\dot{\theta}\dot{\phi}\sin\theta$$
$$= -A\ddot{\phi}\cos\theta, \quad \text{practically,}$$
$$\frac{dH_{y'}}{dt} = A\ddot{\theta},$$
$$\frac{dH_{z'}}{dt} = 0.$$

Substituting into equations (16·4) the quantities found above, then neglecting all terms containing either $\dot{\phi}^2$ or $\dot{\theta}\dot{\phi}$, and putting $\cos\theta = 1$, we get

$$\left.\begin{aligned} -A\ddot{\phi} + C\omega\dot{\theta} &= M_{x'}, \\ A\ddot{\theta} + C\omega\dot{\phi} &= M_{y'}. \end{aligned}\right\} \qquad (68{\cdot}1)$$

To find $M_{x'}$ and $M_{y'}$, let W denote the total weight of gyroscope and mountings, and let l denote the distance from O to the c.g. of the pendulum. Then since ϕ and θ are assumed to be small angles, the moment arms of W about the x- and y-axes are $l\phi$ and $l\theta$, respectively. (The reader is reminded that in Fig. 53 the pendulum is in the octant diametrically opposite the first octant.)

The components of W parallel and perpendicular to OX' are $W\sin\theta$ and $W\cos\theta$, respectively; and the components parallel and perpendicular to OY' are $W\sin\phi$ and $W\cos\phi$, respectively. Only the perpendicular components $W\cos\theta$ and $W\cos\phi$ have moments about the two axes mentioned.

Since OY' and OY are in the same vertical plane, the moment arm of $W\cos\phi$ about OY' is $l\theta$. The moment arm of $W\cos\theta$ about OX' is $l\phi + l\theta\tan\theta\sin\phi$, or $l\phi$ practically.

Now bearing in mind the convention as to the signs of moments, we have

$$M_{x'} = (l\phi)\,W\cos\theta = Wl\phi, \quad \text{practically,}$$
$$M_{y'} = -(l\theta)\,W\cos\phi = -Wl\theta, \quad \text{practically.}$$

Substituting these into equations (68·1), we get

$$\left.\begin{aligned} &(a) \quad A\ddot{\phi} - C\omega\dot{\theta} + Wl\phi = 0, \\ &(b) \quad A\ddot{\theta} + C\omega\dot{\phi} + Wl\theta = 0. \end{aligned}\right\} \qquad (68{\cdot}2)$$

These equations determine the subsequent motion of the pendulum when it is slightly disturbed from its equilibrium position.

69. Solution of the equations

To solve equations (68·2), we eliminate one of the variables and thereby reduce the system to a single fourth-order equation in the other. To eliminate θ, for example, we first differentiate (b) with respect to t and then eliminate $\dot{\theta}$ between the resulting equation and (a), thus obtaining

$$(c)\quad A\ddot{\theta}+C\omega\dddot{\phi}+\frac{Wl}{C\omega}(A\ddot{\phi}+Wl\phi)=0.$$

We next differentiate (a) twice with respect to t and eliminate $\dddot{\theta}$ between the resulting equation and (c), thereby obtaining

$$A^2\phi^{\text{iv}}+(C^2\omega^2+2AWl)\,\ddot{\phi}+W^2l^2\phi=0. \qquad (69\cdot1)$$

The same procedure will give a similar equation in θ and its derivatives.

The auxiliary equation for (69·1) is

$$A^2r^4+(C^2\omega^2+2AWl)\,r^2+W^2l^2=0. \qquad (69\cdot1a)$$

Hence

$$r^2=\frac{-(C^2\omega^2+2AWl)\pm\sqrt{\{(C^2\omega^2+2AWl)^2-4A^2W^2l^2\}}}{2A^2}$$

$$=\frac{-(C^2\omega^2+2AWl)\pm C\omega\sqrt{(C^2\omega^2+4AWl)}}{2A^2}.$$

Since the radical in the above expressions is positive, the two values of r^2 are both real; and they are both negative, because the radical is less than $C^2\omega^2+2AWl$. Let these values of r^2 be $-k_1^2$ and $-k_2^2$. Then

$$r_1^2=-k_1^2=\frac{-(C^2\omega^2+2AWl)+C\omega\sqrt{(C^2\omega^2+4AWl)}}{2A^2}$$

$$r_2^2=-k_2^2=\frac{-(C^2\omega^2+2AWl)-C\omega\sqrt{(C^2\omega^2+4AWl)}}{2A^2}.$$

Hence $$r_1=\pm ik_1,\quad r_2=\pm ik_2,$$

and $$k_1^2=\frac{C^2\omega^2+2AWl-C\omega\sqrt{(C^2\omega^2+4AWl)}}{2A^2} \qquad (69\cdot2)$$

$$k_2^2=\frac{C^2\omega^2+2AWl+C\omega\sqrt{(C^2\omega^2+4AWl)}}{2A^2}. \qquad (69\cdot3)$$

Hence
$$k_2^2+k_1^2=\frac{C^2\omega^2+2AWl}{A^2} \tag{69·4}$$

$$\left.\begin{aligned} k_2^2-k_1^2&=\frac{C\omega\sqrt{(C^2\omega^2+4AWl)}}{A^2},\\ k_2^2k_1^2&=\frac{W^2l^2}{A^2},\end{aligned}\right\} \tag{69·5}$$

or
$$k_2k_1=\frac{Wl}{A}. \tag{69·6}$$

From (69·4), (69·5) and (69·6) we find by elementary algebra

$$k_2=\frac{C\omega+\sqrt{(C^2\omega^2+4AWl)}}{2A}, \tag{69·7}$$

$$k_1=\frac{-C\omega+\sqrt{(C^2\omega^2+4AWl)}}{2A}. \tag{69·8}$$

Hence
$$k_2+k_1=\frac{\sqrt{(C^2\omega^2+4AWl)}}{A}, \tag{69·8a}$$

$$k_2-k_1=\frac{C\omega}{A}. \tag{69·8b}$$

The values of k_2 and k_1 can be reduced to simpler forms as follows:

$$k_2=\frac{C\omega+C\omega\sqrt{\left(1+\frac{4AWl}{C^2\omega^2}\right)}}{2A}=\frac{C\omega}{2A}\left[1+\left(1+\frac{4AWl}{C^2\omega^2}\right)^{\frac{1}{2}}\right].$$

Expanding the radical into a binomial series and retaining only the first two terms of the expansion, we have

$$k_2=\frac{C\omega}{2A}\left[1+1+\frac{2AWl}{C^2\omega^2}+\dots\right]=\frac{C\omega}{A}\left(1+\frac{AWl}{C^2\omega^2}\right).$$

A gyroscopic spherical pendulum may be of any length, but when used as a stabilizer it must have a long period of precession and this necessitates that the quantity $AWl/C^2\omega^2$ be negligible in comparison with 1. This in turn necessitates that l be as small as practicable. On neglecting $AWl/C^2\omega^2$ in the above value for k_2, we get

$$k_2=\frac{C\omega}{A}. \tag{69·9}$$

Proceeding in a similar manner with (69·8), we have

$$k_1=\frac{C\omega}{2A}\left[-1+\left(1+\frac{4AWl}{C^2\omega^2}\right)^{\frac{1}{2}}\right]=\frac{C\omega}{2A}\left[-1+1+\frac{2AWl}{C^2\omega^2}\right].$$

Here we are not justified in neglecting $AWl/C^2\omega^2$ in comparison with zero. Hence we retain it and get

$$k_1=\frac{Wl}{C\omega}. \tag{69·10}$$

A glance at these values of k_1 and k_2 shows that when ω is large (as it always is in stabilizers) k_2 is large and k_1 is very small. Hence k_1 is negligible in comparison with k_2.

Since the values of r are $\pm ik_1$ and $\pm ik_2$, the solution of equation (69·1) is

$$\phi=c_1\sin k_1t+c_2\cos k_1t+c_3\sin k_2t+c_4\cos k_2t. \tag{69·11}$$

To find θ without bringing in additional constants of integration, we first differentiate (68·2 (a)) with respect to t and then eliminate $\ddot{\theta}$ between the resulting equation and (68·2 (b)), thereby obtaining

$$\theta=-\frac{1}{C\omega Wl}[A^2\dddot{\phi}+(C^2\omega^2+AWl)\dot{\phi}]. \tag{69·11a}$$

Then we find the values of $\dot{\phi}$ and $\dddot{\phi}$ from (69·11) and substitute them into the above equation, thus obtaining

$$\begin{aligned}\theta=-\frac{1}{C\omega Wl}\{&c_2k_1[A^2k_1^2-(C^2\omega^2+AWl)]\sin k_1t\\ &-c_1k_1[A^2k_1^2-(C^2\omega^2+AWl)]\cos k_1t\\ &+c_4k_2[A^2k_2^2-(C^2\omega^2+AWl)]\sin k_2t\\ &-c_3k_2[A^2k_2^2-(C^2\omega^2+AWl)]\cos k_2t\}.\end{aligned} \tag{69·12}$$

To simplify the coefficients of the trigonometric terms we replace k_1^2 and k_1 by their values from (69·2) and (69·8), respectively, and thus get

$$A^2k_1^2-(C^2\omega^2+AWl)=-\frac{C\omega}{2}[C\omega+\sqrt{(C^2\omega^2+4AWl)}],$$

and $$k_1[A^2k_1^2-(C^2\omega^2+AWl)]=-C\omega Wl.$$

From (69·3) and (69·7) we find in a similar manner that

$$k_2[A^2k_2^2-(C^2\omega^2+AWl)]=C\omega Wl.$$

On substituting into (69·12) these values of the coefficients, we get

$$\theta=c_2\sin k_1t-c_1\cos k_1t-c_4\sin k_2t+c_3\cos k_2t. \tag{69·13}$$

Equations (69·11) and (69·13) each represent two simultaneous and superposed simple harmonic motions of periods

$$T_1 = \frac{2\pi}{k_1} = 2\pi \frac{C\omega}{Wl}, \quad \text{the precession period,}$$

and
$$T_2 = \frac{2\pi}{k_2} = 2\pi \frac{A}{C\omega}, \quad \text{the nutation period.}$$

T_1 is thus a long period and T_2 a very short period. The motion for either ϕ or θ is therefore a pseudo-regular precession as discussed in Art. 38.

If in (69·11) we decrease t in the k_1-terms by a quarter period $\left(\frac{1}{4}\frac{2\pi}{k_1}\right)$, and increase t in the k_2-terms by a quarter period $\left(\frac{1}{4}\frac{2\pi}{k_2}\right)$, the right-hand member becomes

$$-c_1 \cos k_1 t + c_2 \sin k_1 t + c_3 \cos k_2 t - c_4 \sin k_2 t,$$

which is the value of θ as given in (69·13). This result shows that in the long-period motion the phase of θ is a quadrant behind ϕ and in the short-period motion it is a quadrant ahead of ϕ.

We now determine the constants of integration from assumed initial conditions. From (69·11) and (69·13) we get

$$\dot{\phi} = c_1 k_1 \cos k_1 t - c_2 k_1 \sin k_1 t + c_3 k_2 \cos k_2 t - c_4 k_2 \sin k_2 t, \tag{69·14}$$

$$\dot{\theta} = c_2 k_1 \cos k_1 t + c_1 k_1 \sin k_1 t - c_4 k_2 \cos k_2 t - c_3 k_2 \sin k_2 t. \tag{69·15}$$

Let us assume that while the pendulum was hanging in equilibrium vertically downward, with gyro spinning at constant speed ω, it was struck a horizontal blow which imparted to it an angular velocity $\dot{\theta}_0$ in the direction of the negative x-axis. Then the initial conditions are

$$\phi = 0, \quad \theta = 0, \quad \dot{\phi} = 0, \quad \dot{\theta} = \dot{\theta}_0 \quad \text{when} \quad t = 0.$$

On substituting these values into (69·11), (69·13), (69·14) and (69·15), we find that

$$c_1 = 0, \quad c_2 = \frac{\dot{\theta}_0}{k_1 + k_2}, \quad c_3 = 0, \quad c_4 = -\frac{\dot{\theta}_0}{k_1 + k_2}.$$

Hence equations (69·11) and (69·13) become

$$\phi=\frac{\dot\theta_0}{k_1+k_2}(\cos k_1t-\cos k_2t),$$

$$\theta=\frac{\dot\theta_0}{k_1+k_2}(\sin k_1t+\sin k_2t).$$

Now neglecting k_1 in comparison with k_2 and replacing k_2 by its value $c\omega/A$ from (69·9), we have

$$\phi=\frac{A\dot\theta_0}{C\omega}(\cos k_1t-\cos k_2t), \tag{69·16}$$

$$\theta=\frac{A\dot\theta_0}{C\omega}(\sin k_1t+\sin k_2t). \tag{69·17}$$

These equations show that the disturbed motion is stable and that the amplitude is very small when ω is large.

It is of some interest to determine the constants of integration in (69·11) and (69·13) for a different set of initial conditions. Let us assume that the pendulum is swinging in the plane $\phi=0$ and is struck a sharp blow at right angles to its plane of vibration at the instant it reaches the end of its swing. If the blow imparts to the pendulum an angular velocity $\dot\phi_0$ in the direction of the negative y-axis, the initial conditions are

$$\phi=0,\quad \dot\phi=\dot\phi_0,\quad \theta=\theta_0,\quad \dot\theta_0=0,\quad \text{when}\quad t=0.$$

Substituting these quantities into equations (69·11), (69·13), (69·14), (69·15), we find

$$c_1=\frac{\dot\phi_0-k_2\theta_0}{k_1+k_2},\quad c_2=0,\quad c_3=\frac{\dot\phi_0+k_1\theta_0}{k_1+k_2},\quad c_4=0.$$

Then equations (69·11) and (69·13) become

$$\phi=\frac{\dot\phi_0-k_2\theta_0}{k_1+k_2}\sin k_1t+\frac{\dot\phi_0+k_1\theta_0}{k_1+k_2}\sin k_2t,$$

$$\theta=-\frac{\dot\phi_0-k_2\theta_0}{k_1+k_2}\cos k_1t+\frac{\dot\phi_0+k_1\theta_0}{k_1+k_2}\cos k_2t.$$

Since k_1 and θ_0 are both small, their product is negligible in

comparison with $\dot{\phi}_0$; and since k_1 is negligible in comparison with k_2, the above equations become

$$\left.\begin{aligned}\phi&=\left(-\theta_0+\frac{\dot{\phi}_0}{k_2}\right)\sin k_1 t+\frac{\dot{\phi}_0}{k_2}\sin k_2 t,\\ \theta&=\left(\theta_0-\frac{\dot{\phi}_0}{k_2}\right)\cos k_1 t+\frac{\dot{\phi}_0}{k_2}\cos k_2 t.\end{aligned}\right\} \tag{69·18}$$

These equations can be simplified still further in view of the fact that $\dot{\phi}_0/k_2$ is negligible in comparison with θ_0. Hence we neglect this fraction in equations (69·18) and get

$$\left.\begin{aligned}\phi&=-\theta_0\sin k_1 t,\\ \theta&=\theta_0\cos k_1 t.\end{aligned}\right\} \tag{69·19}$$

These equations show that in this case the motion of the pendulum is a slow precession in a circle of radius $l\theta_0$.

70. The spherical pendulum without gyroscope

If we put $\omega=0$ in equations (68·2), we get the differential equations of motion of an ordinary spherical pendulum. We thus have

$$\left.\begin{aligned}A\ddot{\phi}+Wl\phi&=0,\\ A\ddot{\theta}+Wl\theta&=0.\end{aligned}\right\} \tag{70·1}$$

The auxiliary equation for either of these equations is

$$Ar^2+Wl=0,$$

from which $$r=\pm i\sqrt{\frac{Wl}{A}}=\pm ki, \quad \text{say.}$$

Hence the solutions are

(*a*) $\phi=c_1\sin kt+c_2\cos kt$,

(*b*) $\theta=c_3\sin kt+c_4\cos kt$.

Then

(*c*) $\dot{\phi}=c_1 k\cos kt-c_2 k\sin kt$,

(*d*) $\dot{\theta}=c_3 k\cos kt-c_4\sin kt$.

For the initial conditions

$$\phi=0,\quad \theta=0,\quad \dot{\phi}=0,\quad \dot{\theta}=\dot{\theta}_0 \quad \text{when} \quad t=0,$$

we find
$$c_1=0,\quad c_2=0,\quad c_3=\dot{\theta}_0 k,\quad c_4=0.$$

Hence the equations of motion are

$$\left.\begin{aligned}\phi&=0,\\ \theta&=\dot{\theta}_0\sqrt{\left(\frac{A}{Wl}\right)}\sin\sqrt{\left(\frac{Wl}{A}\right)}\,t.\end{aligned}\right\} \tag{70·2}$$

The pendulum thus vibrates with simple harmonic motion in the plane $\phi=0$. It will be noted that the amplitude is here much larger than in (69·17), because of the presence of ω in the denominator of the amplitude of the latter.

For the initial conditions

$$\phi=0,\quad \dot{\phi}=\dot{\phi}_0,\quad \theta=\theta_0,\quad \dot{\theta}=0 \quad \text{when } t=0,$$

we find from (*a*), (*b*), (*c*), (*d*),

$$c_1=\dot{\phi}_0\sqrt{\left(\frac{A}{Wl}\right)},\quad c_2=0,\quad c_3=0,\quad c_4=\theta_0.$$

Hence equations (*a*) and (*b*) become

$$\left.\begin{aligned}\phi&=\dot{\phi}_0\sqrt{\left(\frac{A}{Wl}\right)}\sin\sqrt{\left(\frac{Wl}{A}\right)}\,t,\\ \theta&=\theta_0\cos\sqrt{\left(\frac{Wl}{A}\right)}\,t.\end{aligned}\right\} \tag{70·3}$$

On multiplying these equations through by l and recalling that $l\phi=y$ and $l\theta=x$, we have

$$y=l\dot{\phi}_0\sqrt{\left(\frac{A}{Wl}\right)}\sin\sqrt{\left(\frac{Wl}{A}\right)}\,t,$$

$$x=l\theta_0\cos\sqrt{\left(\frac{Wl}{A}\right)}\,t,$$

which are the parametric equations of an ellipse with semiaxes

$$l\dot{\phi}_0\sqrt{\left(\frac{A}{Wl}\right)} \quad \text{and} \quad l\theta_0.$$

The period of revolution is

$$T = 2\pi \sqrt{\left(\frac{A}{Wl}\right)}.$$

In the gyroscopic pendulum the long period is $2\pi\frac{C\omega}{Wl}$, which is very much longer because of the factor ω.

71. Numerical comparison of the amplitudes of the gyroscopic and spherical pendulums

As a numerical case we consider a gyro or rotor 8 inches in diameter, having a rim 3 inches wide and 2 inches thick, and a web $\frac{1}{2}$ inch thick. Assuming that the distance from the point of support to the c.g. of the pendulum is 3 inches, that steel weighs 490 pounds per cubic foot, and that the rotor is spinning at 10,000 r.p.m., we have the following data:

$$l = \tfrac{1}{4} \text{ foot},$$

$$W = 33.8 \text{ pounds},$$

$$\omega = 10{,}000 \text{ r.p.m.} = 838 \text{ radians per second},$$

$$C = 0.070, \text{ in pound-feet-seconds units},$$

$$A = 0.040 + \frac{33.8}{32.2}\left(\frac{1}{4}\right)^2 = 0.106, \text{ in pound-feet-seconds units.}$$

Then

$$k_2 = \frac{C\omega}{A} = \frac{58.7}{0.106} = 554,$$

$$k_1 = \frac{Wl}{C\omega} = \frac{33.8 \times \frac{1}{4}}{58.7} = 0.144,$$

$$T_2 = \frac{2\pi}{554} = 0.011 \text{ second},$$

$$T_1 = \frac{2\pi}{0.144} = 43.6 \text{ seconds}.$$

It will be recalled that the equations of motion for both the gyroscopic pendulum and the spherical pendulum were derived on the assumption that ϕ and θ were to be small angles. Hence the assumed value of $\dot{\theta}_0$ must be such that ϕ and θ for both types of

pendulum will not exceed 3°, say. A preliminary calculation shows that $\dot{\theta}_0$ must not exceed $\frac{1}{2}$ radian per second. Hence we take $\dot{\theta}_0 = \frac{1}{2}$ radian per second, and substitute this value in equations (69·16), (69·17) and (70·2). From (69·16) and (69·17):

$$\phi = \frac{\frac{1}{2}(0.106)}{58.7}(\cos 0.144t - \cos 554t)$$

$$= 0.00090(\cos 0.144t - \cos 554t),$$

$$\theta = 0.00090(\sin 0.144t + \sin 554t).$$

Since the algebraic sum of the trigonometric terms in parentheses can never exceed 2, we have

$$\phi_{\text{max.}} = \theta_{\text{max.}} \lesseqgtr 0.0018 \text{ radian} = 6', \quad \text{approximately.}$$

In the case of the ordinary spherical pendulum we have, from (70·2),

$$\theta_{\text{max.}} = \tfrac{1}{2}\sqrt{\frac{0.106}{33.8 \times \frac{1}{4}}} = 0.056 \text{ radian} = 3°.2 = 192'.$$

The amplitude in this case is thus 32 times as great as in the gyroscopic pendulum under the same initial conditions. The great stability of the gyroscopic pendulum makes it of great value as a stabilizer.

The period of this spherical pendulum is

$$T = 2\pi\sqrt{\left(\frac{A}{Wl}\right)} = 2\pi\sqrt{\left(\frac{0.106}{33.8 \times \frac{1}{4}}\right)} = 0.112 \text{ second},$$

or about ten times as great as the short (nutation) period of the gyroscopic pendulum.

72. Gyroscopic pendulum with point of suspension subjected to a periodic force

When a gyroscopic pendulum is mounted on a ship, the rolling of the ship about its longitudinal axis subjects the point of suspension to a to-and-fro motion athwartship. This motion is periodic and produces a forced vibration of the pendulum.

To investigate the stability of the motion of the gyropendulum in this case, we reduce the problem to that of the nonrolling case by applying d'Alembert's principle (Art. 11) and then proceed as in Art. 68. In the case of the gyropendulum the external applied

force is unknown, but the inertia force is $-Wa/g$ and acts through the center of gravity of the pendulum. It thus has a moment about any axis not parallel to it and not intersecting it.

Referring to Fig. 53, let us assume that OX is parallel to the longitudinal axis of the ship. Then the periodic motion of the point of suspension of the pendulum is in the direction of OY and may be represented by a sine function of the time. Hence we assume that the motion is represented by the equation

$$y = a \sin bt.$$

Then

$$\dot{y} = ab \cos bt,$$

$$\ddot{y} = -ab^2 \sin bt.$$

The inertia force is therefore

$$F = -\frac{W}{g}\ddot{y} = \frac{W}{g} ab^2 \sin bt,$$

and acts to the right parallel to OY. Hence for the moment of this force we have

$$M_{x'} = Fl = \frac{W}{g} ab^2 l \sin bt,$$

$$M_{y'} = 0, \quad M_{z'} = 0.$$

Adding these moments to the gravity moments of Art. 68, we get

$$\left.\begin{aligned} &(a) \quad A\ddot{\phi} - C\omega\dot{\theta} + Wl\phi = \frac{W}{g} ab^2 l \sin bt, \\ &(b) \quad A\ddot{\theta} + C\omega\dot{\phi} + Wl\theta = 0, \end{aligned}\right\} \tag{72·1}$$

as the differential equations of motion in this case.

On eliminating θ as in Art. 69, we get

$$A^2\phi^{\text{iv}} + (C^2\omega^2 + 2AWl)\ddot{\phi} + W^2l^2\phi = \frac{Wab^2l(Wl - Ab^2)}{g} \sin bt. \tag{72·2}$$

To find a particular solution of (72·2) we put

$$\phi = P \sin bt + Q \cos bt, \tag{72·3}$$

where P and Q are constants to be determined. Then

$$\ddot{\phi} = -Pb^2 \sin bt - Qb^2 \cos bt,$$

$$\phi^{\text{iv}} = Pb^4 \sin bt + Qb^4 \cos bt.$$

Substituting into (72·2) these values of ϕ, $\ddot{\phi}$ and ϕ^{iv} and then equating the coefficients of $\sin bt$ and $\cos bt$ on the two sides of the resulting equation, we get

$$(c)\quad PA^2b^4-(C^2\omega^2+2AWl)Pb^2+PW^2l^2=\frac{Wab^2l(Wl-Ab^2)}{g},$$

$$(d)\quad QA^2b^4-(C^2\omega^2+2AWl)Qb^2+QW^2l^2=0.$$

From (c) we get

$$(e)\quad P=\frac{Walb^2(Wl-Ab^2)}{g[A^2b^4-(C^2\omega^2+2AWl)b^2+W^2l^2]};$$

and from (d) we have

$$(f)\quad Q[A^2b^4-(C^2\omega^2+2AWl)b^2+W^2l^2]=0.$$

Here either Q or the expression in brackets must be zero. We first examine the bracketed expression.

On putting $r^2=-k^2$ in (69·1 a), we get

$$A^2k^4-(C^2\omega^2+2AWl)k^2+W^2l^2=0,$$

which is the bracketed expression in (f) if b is replaced by k. Hence the bracketed expression cannot be zero unless $b=k$, which means that $b_1=k_1$ and $b_2=k_2$. But since the rolling period of a ship is not as slow as the period given by k_1 nor as fast as the period given by k_2, we must conclude that the expression in brackets is not zero. Hence we must have

$$Q=0.$$

Substituting into (72·3) the values of P and Q just found, we have

$$\phi=\frac{Walb^2(Wl-Ab^2)}{g[A^2b^4-(C^2\omega^2+2AWl)b^2+W^2l^2]}\sin bt, \tag{72·4}$$

as a particular solution of (72·1) for ϕ.

To find the corresponding particular solution for θ, we differentiate (72·1 (a)) with respect to t and then eliminate $\ddot{\theta}$ between the resulting equation and (72·1 (b)). We thus find

$$\theta=-\frac{1}{C\omega Wl}\left[A^2\dddot{\phi}+(C^2\omega^2+AWl)\dot{\phi}-\frac{WAalb^3}{g}\cos bt\right]. \tag{72·5}$$

We next find $\dot{\phi}$ and $\dddot{\phi}$ from (72·4) and substitute their values into (72·5), thus obtaining

$$\theta=-\frac{C\omega Walb^3}{g[A^2b^4-(C^2\omega^2+2AWl)b^2+W^2l^2]}\cos bt. \tag{72·6}$$

Now since

$$A^2b^4-(C^2\omega^2+2AWl)b^2+W^2l^2=-[C^2\omega^2b^2-(Wl-Ab^2)^2],$$

we can write (72·4) and (72·6) in the simpler forms

$$\phi=-\frac{Walb^2(Wl-Ab^2)}{g[C^2\omega^2b^2-(Wl-Ab^2)^2]}\sin bt, \tag{72·7}$$

$$\theta=\frac{C\omega Walb^3}{g[C^2\omega^2b^2-(Wl-Ab^2)^2]}\cos bt. \tag{72·8}$$

These equations show that the rolling of the ship produces forced vibrations of the pendulum not only athwart the ship but also in the direction of the longitudinal axis. The presence of ω in the numerator of (72·8) also shows that the amplitude of θ is much greater than that of ϕ.

On replacing bt by $bt-\frac{1}{2}\pi$, we find that (72·7) takes the form of (72·8), which shows that the θ-vibration is a quarter period behind the ϕ-vibration.

Because of the presence of ω^2 in the denominators of (72·7) and (72·8) the amplitudes of the forced vibrations are very small. If we put $\omega=0$, the equations become

$$\left.\begin{aligned}\phi&=\frac{Walb^2}{g(Wl-Ab^2)}\sin bt,\\ \theta&=0,\end{aligned}\right\} \tag{72·9}$$

which show that in this case the pendulum vibrates only in the plane perpendicular to the axis of roll.

When the gyro spins at very high speed, as it always does in stabilizers, the quantity $(Wl-Ab^2)^2$ is negligible in comparison with $C^2\omega^2b^2$. Hence on neglecting $(Wl-Ab^2)^2$ in the denominators of (72·7) and (72·8), we get

$$\phi=-\frac{Wal(Wl-Ab^2)}{gC^2\omega^2}\sin bt, \tag{72·10}$$

$$\theta=\frac{Walb}{gC\omega}\cos bt. \tag{72·11}$$

These equations show at a glance that the amplitudes of the forced vibrations of a gyroscopic pendulum are very small, and by comparison with (72·9) they are seen to be much smaller than the

forced vibrations of the same pendulum with the gyro not spinning. Hence the spin axis of the gyroscopic pendulum remains practically vertical in spite of the rolling of the ship.

Note. The condition that a gyropendulum be unaffected by the motion of the ship or airplane on which it is mounted is that $C\omega/Wl = \sqrt{(R/g)}$, where R denotes the radius of the earth. For the derivation of this condition, see the Appendix.

An important application of the gyroscopic pendulum is in the automatic firing of naval guns on a rolling and pitching ship. The gyroscopic pendulum maintains a horizon of reference. The aiming device of the gun is set at the required elevation above the horizon for hitting the distant target. When the roll of the ship brings the axis of the gun barrel to the preset angle of elevation, electric contacts close a circuit that fires the gun. In case the roll of the ship is not sufficient to raise the gun to the required elevation above the horizon, the preset angle of elevation is made great enough to allow for the insufficient roll.

B. OTHER TYPES OF GYROSCOPIC PENDULUMS

73. Gyroscopic pendulum suspended at center of gravity of gyro and with pendulous weight below

Gyroscopic pendulums may take several forms. A common type is that illustrated in Fig. 54. It will be seen that the gyro is suspended in Cardan rings, the outer ring resting on pivots A and A' attached to supports not shown in the figure. The pivot axes AA' and BB' are perpendicular to each other and intersect at the center of gravity of the gyro. The plane of the gyro is kept approximately horizontal by means of a pendulous weight W.

The mounting permits the pendulum to swing through an angle ϕ about the axis AA' and through an angle θ about the axis BB'. Fig. 55 shows the position of the pendulum and pivot axes after the pendulum has been disturbed from its position of equilibrium. (In this figure the pivot axes of Fig. 54 have been rotated 90° to the right.)

Comparison of Fig. 55 with Fig. 53 shows that the equations of motion in this case are exactly the same as those of the gyroscopic spherical pendulum found in Arts. 68 and 69. The derivation of the equations of motion will therefore not be repeated here.

In a numerical example, however, there is one important difference between the gyroscopic pendulum of the present article and the gyroscopic spherical pendulum previously treated. That difference is in the transverse moment of inertia A. In the case of the gyroscopic spherical pendulum, A must be found by means of the parallel-axis theorem relating to moments of inertia, namely,

$$A_L = A_G + Md^2,$$

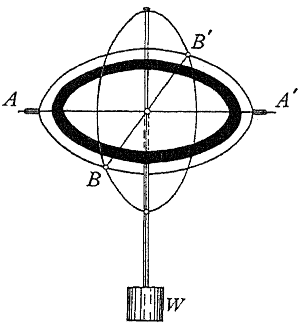


Fig. 54

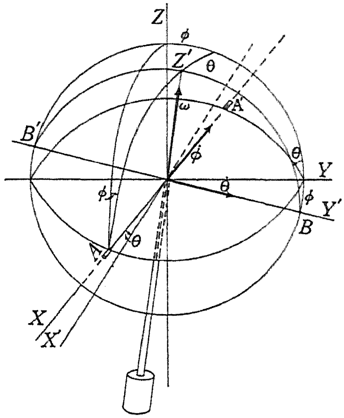


Fig. 55

where A_G denotes the transverse moment of inertia of a body of mass M about an axis through its center of gravity, and A_L is the moment of inertia about any other axis L parallel to the gravity axis and distant d from it. The above formula was used in computing A in the example of Art. 71. In the present case, A is simply A_G.

Gyroscopic pendulums mounted as in Fig. 54 are used in stabilizing ships and airplanes.

74. The inverted gyroscopic pendulum

Gyroscopic pendulums with pendulous weight above the point of support have application in monorail cars. Fig. 56 shows such a pendulum when displaced from a vertical position. The gyro may be mounted as in Fig. 54 with its c.g. at the point of support, or it may constitute the pendulum itself as in the case of the gyroscopic spherical pendulum. The mathematics of the two cases is the same except for the transverse moment of inertia A.

From Fig. 54 we have

$$\omega_{x'} = -\dot{\phi}\cos\theta, \quad \omega_{y'} = \dot{\theta}, \quad \omega_{z'} = -\dot{\phi}\sin\theta.$$

Then $H_{x'} = -A\dot{\phi}\cos\theta$, $H_{y'} = A\dot{\theta}$, $H_{z'} = C(\omega - \dot{\phi}\sin\theta)$

$= C\omega$, practically.

Hence

$$\frac{dH_{x'}}{dt} = -A(\ddot{\phi}\cos\theta - \dot{\theta}\dot{\phi}\sin\theta) = -A\ddot{\phi}\cos\theta, \quad \text{practically,}$$

$$\frac{dH_{y'}}{dt} = A\ddot{\theta}, \quad \frac{dH_{z'}}{dt} = 0.$$

Also, $$M_{x'} = -Wl\phi, \quad M_{y'} = Wl\theta.$$

Substituting the above quantities into (16·4), we get

$$-A\ddot{\phi}\cos\theta + C\omega\dot{\theta} + A\dot{\theta}\dot{\phi}\sin\theta = -Wl\phi,$$

$$A\ddot{\theta} + A\dot{\phi}^2\sin\theta\cos\theta + C\omega\dot{\phi}\cos\theta = Wl\theta,$$

$$-A\dot{\theta}\dot{\phi}\cos\theta + A\dot{\theta}\dot{\phi}\cos\theta = 0.$$

Neglecting the terms which contain $\dot{\phi}^2$ and $\dot{\theta}\dot{\phi}$ and putting $\cos\theta = 1$, we get

$$\left.\begin{aligned} &(a) \quad A\ddot{\phi} - C\omega\dot{\theta} = Wl\phi, \\ &(b) \quad A\ddot{\theta} + C\omega\dot{\phi} = Wl\theta. \end{aligned}\right\} \tag{74·1}$$

It will be seen that these equations differ from (68·2) only in the sign of l.

To solve equations (74·1) it is not necessary to go through the process of eliminating θ. To get the fourth-order equation in ϕ that results from such elimination, we simply change the sign of l in equation (69·1). We thus get

$$A^2\phi^{\text{iv}} + (C^2\omega^2 - 2AWl)\ddot{\phi} + W^2l^2\phi = 0. \tag{74·2}$$

An exactly similar equation holds for θ.

The auxiliary equation for either equation is

$$A^2r^4 + (C^2\omega^2 - 2AWl)r^2 + W^2l^2 = 0,$$

from which

$$r^2 = \frac{-(C^2\omega^2 - 2AWl) \pm \sqrt{\{(C^2\omega^2 - 2AWl)^2 - 4A^2W^2l^2\}}}{2A^2}$$

$$= \frac{-(C^2\omega^2 - 2AWl) \pm C\omega\sqrt{(C^2\omega^2 - 4AWl)}}{2A^2}.$$

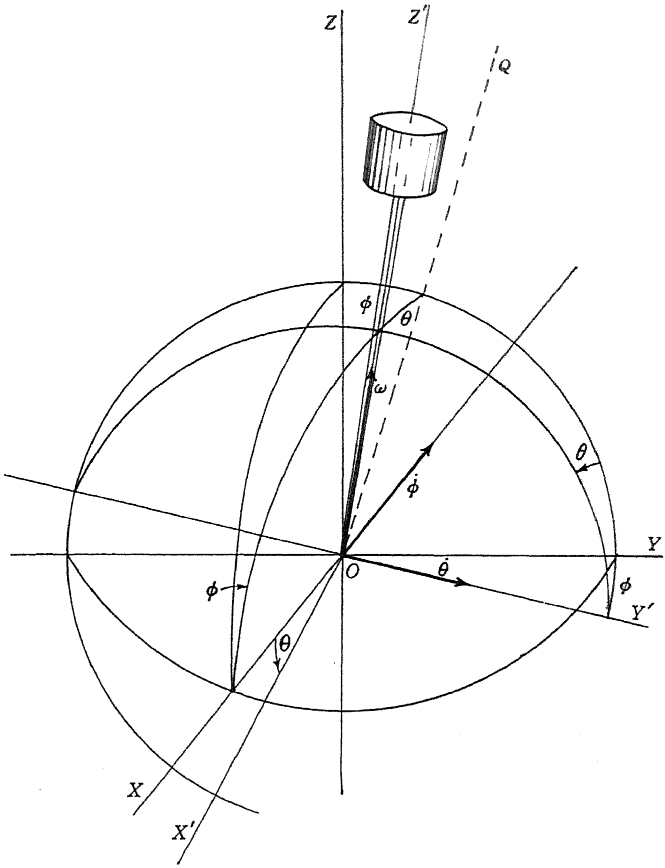


Fig. 56

Since both values of r^2 are real and negative when $C^2\omega^2 > 4AWl$, we write

$$r_1^2 = -k_1^2, \quad r_2^2 = -k_2^2.$$

Then
$$k_1^2 = \frac{C^2\omega^2 - 2AWl - C\omega\sqrt{(C^2\omega^2 - 4AWl)}}{2A^2},$$

$$k_2^2 = \frac{C^2\omega^2 - 2AWl + C\omega\sqrt{(C^2\omega^2 - 4AWl)}}{2A^2}.$$

From these last two equations we get, by the same procedure as in Art. 69,

$$k_2 = \frac{C\omega + \sqrt{(C^2\omega^2 - 4AWl)}}{2A},$$

$$k_1 = \frac{C\omega - \sqrt{(C^2\omega^2 - 4AWl)}}{2A}.$$

By replacing the radicals by the first two terms of their binomial expansions, we get the simpler results:

$$k_2 = \frac{C\omega\left[1 + \left(1 - \frac{4AWl}{C^2\omega^2}\right)^{\frac{1}{2}}\right]}{2A} = \frac{C\omega}{2A}\left(1 + 1 - \frac{2AWl}{C^2\omega^2}\right)$$

$$= \frac{C\omega}{A},$$

$$k_1 = \frac{C\omega}{2A}\left[1 - \left(1 - \frac{4AWl}{C^2\omega^2}\right)^{\frac{1}{2}}\right] = \frac{Wl}{C\omega}.$$

The solution of (73·2) is therefore

$$\phi = c_1 \sin k_1 t + c_2 \cos k_1 t + c_3 \sin k_2 t + c_4 \cos k_2 t. \qquad (74{\cdot}3)$$

An exactly similar equation holds for θ, except for the signs and positions of the c's (see Art. 69).

We could determine the integration constants in (74·3) by the same procedure as in Art. 69, but it seems unnecessary to do so. The important result we have arrived at is that, since the values of ϕ and θ do not increase with time, the motion of the inverted gyroscopic pendulum is stable when the rotor spins fast enough to satisfy the condition $C^2\omega^2 > 4AWl$.

75. Gyroscopic pendulum with gimbal axes rotating in azimuth

When a gyroscopic pendulum is mounted on board ship or on an airplane and the ship or airplane turns in azimuth, the gyroscopic pendulum turns with the ship or airplane. Moreover, gyroscopic devices are sometimes mounted on platforms which can be rotated in azimuth in either direction with respect to the deck of the ship. Hence it is desirable to investigate the behavior of a gyroscopic pendulum when it is compelled to rotate in azimuth.

Let us consider a gyroscopic pendulum of the type treated in Art. 73 and assume that the gimbal axes AA' and BB' (Fig. 55) are rotating in azimuth with constant angular velocity $\dot{\psi}$ in the same

direction as the gyro is spinning (see Fig. 57). To derive the differential equations of motion in this case, we find the projections of $\dot{\psi}$ on the moving axes, add these to the angular velocities that exist

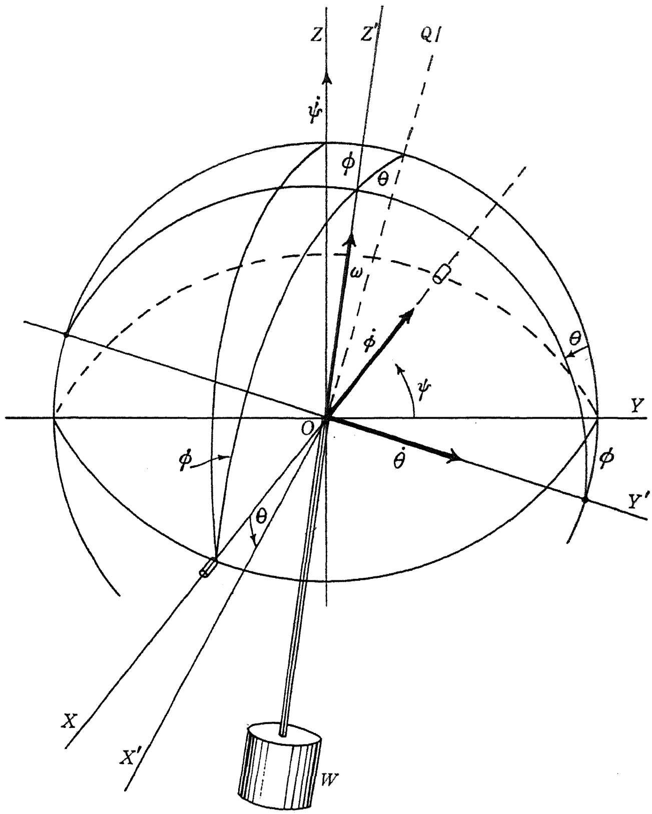


Fig. 57

when there is no rotation, and then use these total angular velocities in connection with equations (16·4). Thus, in Fig. 57 the rectangular components of $\dot{\psi}$ in the YZ-plane are $\dot{\psi}\cos\phi$ along OQ and $-\dot{\psi}\sin\phi$ along OY'. Then $\dot{\psi}\cos\phi$ has the components $\dot{\psi}\cos\phi\cos\theta$ along OZ' and $-\dot{\psi}\cos\phi\sin\theta$ along OX'. Hence the projections of $\dot{\psi}$ on the moving axes are

$$\begin{array}{lll} -\dot{\psi}\cos\phi\sin\theta & \text{on} & OX', \\ -\dot{\psi}\sin\phi & \text{on} & OY', \\ \dot{\psi}\cos\phi\cos\theta & \text{on} & OZ'. \end{array}$$

Since the angular velocities about OX', OY' and OZ' when the gimbals are not rotating are $-\dot{\phi}\cos\theta$, $\dot{\theta}$ and $-\dot{\phi}\sin\theta$, respectively, the total angular velocities about these axes when the gimbals are rotating in azimuth are

$$\omega_{x'} = -\dot{\phi}\cos\theta - \dot{\psi}\cos\phi\sin\theta,$$

$$\omega_{y'} = \dot{\theta} - \dot{\psi}\sin\phi,$$

$$\omega_{z'} = -\dot{\phi}\sin\theta + \dot{\psi}\cos\phi\cos\theta.$$

Hence

$$H_{x'} = -A(\dot{\phi}\cos\theta + \dot{\psi}\cos\phi\sin\theta),$$

$$H_{y'} = A(\dot{\theta} - \dot{\psi}\sin\phi),$$

$$H_{z'} = C(\omega - \dot{\phi}\sin\theta + \dot{\psi}\cos\phi\cos\theta).$$

Then, remembering that $\dot{\psi}$ is a constant,

$$\begin{aligned}\frac{dH_{x'}}{dt} &= -A(\ddot{\phi}\cos\theta - \dot{\phi}\dot{\theta}\sin\theta - \dot{\psi}\dot{\phi}\sin\phi\sin\theta + \dot{\psi}\dot{\theta}\cos\phi\cos\theta)\\ &= -A\ddot{\phi}\cos\theta, \quad \text{approximately,}\end{aligned}$$

$$\frac{dH_{y'}}{dt} = A(\ddot{\theta} - \dot{\psi}\dot{\phi}\cos\phi) = A\ddot{\theta}, \quad \text{approximately,}$$

$$\begin{aligned}\frac{dH_{z'}}{dt} &= C(-\ddot{\phi}\sin\theta - \dot{\phi}\dot{\theta}\cos\theta - \dot{\psi}\dot{\phi}\sin\phi\cos\theta - \dot{\psi}\dot{\theta}\cos\phi\sin\theta)\\ &= -C\ddot{\phi}\sin\theta, \quad \text{approximately}\\ &= 0, \quad \text{practically, since } \theta \text{ is small and } \dot{\phi} \text{ changes slowly.}\end{aligned}$$

On substituting the above quantities into (16·4) and neglecting all terms containing the squares and products of $\dot{\psi}$, $\dot{\theta}$ and $\dot{\phi}$, we get

$$-A\ddot{\phi}\cos\theta + C\omega\dot{\theta} - C\omega\dot{\psi}\sin\phi = M_{x'} = Wl\phi,$$

$$A\ddot{\theta} + C\omega\dot{\phi}\cos\theta + C\omega\dot{\psi}\cos\phi\sin\theta = M_{y'} = -Wl\theta.$$

Now putting $\sin\phi = \phi$, $\sin\theta = \theta$, $\cos\phi = 1$ and $\cos\theta = 1$, we have

$$\left.\begin{aligned}A\ddot{\phi} - C\omega\dot{\theta} + (C\omega\dot{\psi} + Wl)\,\phi = 0,\\ A\ddot{\theta} + C\omega\dot{\phi} + (C\omega\dot{\psi} + Wl)\,\theta = 0.\end{aligned}\right\} \tag{75·1}$$

It will be noted that if $\dot{\psi} = 0$, these equations become identical with (68·2), as they should.

It will be further observed that equations (75·1) are identical in form with (68·2), the only difference being that the constant

coefficient Wl in (68·2) has been replaced by the constant coefficient $C\omega\dot{\psi}+Wl$. Hence we may utilize the solutions of (68·2) to obtain the solutions of (75·1). For example, we have, by equations (69·8a) and (69·8b):

$$k_2+k_1=\frac{\sqrt{\{C^2\omega^2+4A(Wl+C\omega\dot{\psi})\}}}{A}, \tag{75·2}$$

$$k_2-k_1=\frac{C\omega}{A}. \tag{75·3}$$

And k_1 and k_2 in their final reduced forms are, from equations (69·10) and (69·9):

$$k_1=\frac{Wl+C\omega\dot{\psi}}{C\omega},$$

$$k_2=\frac{C\omega}{A}.$$

The periods are therefore

$$T_1=2\pi\frac{C\omega}{Wl+C\omega\dot{\psi}},$$

$$T_2=2\pi\frac{A}{C\omega}.$$

These results show that rotation of the gimbal axes in the same direction as the spin of the gyro makes the long period shorter.

If the gimbals are rotated at constant angular velocity $\dot{\psi}$ in the direction *opposite* the spin of the gyro, the equations of motion are obtained from the preceding by changing the sign of $\dot{\psi}$. The differential equations of motion are then

$$\left.\begin{aligned}A\ddot{\phi}-C\omega\dot{\theta}+(Wl-C\omega\dot{\psi})\,\phi=0,\\ A\ddot{\theta}+C\omega\dot{\phi}+(Wl-C\omega\dot{\psi})\,\theta=0.\end{aligned}\right\} \tag{75·4}$$

The values of k_1, k_2, and the periods in this case are

$$k_1'=\frac{Wl-C\omega\dot{\psi}}{C\omega},$$

$$k_2'=\frac{C\omega}{A},$$

$$T_1'=2\pi\frac{C\omega}{Wl-C\omega\dot{\psi}},$$

$$T_2'=2\pi\frac{A}{C\omega}.$$

Here the long period is longer than is the case when the axes are not rotating.

The reader will note that in the case of opposite rotation here considered, we must have $Wl > C\omega\dot{\psi}$ in order that k_1' and T_1' be positive. This fact puts a restriction on the magnitude of $\dot{\psi}$ if the gyroscopic pendulum is to serve as a stabilizer. Hence when turning in azimuth, the gimbal axes should be turned, if possible, in the same direction as the gyro spins.

The solutions of equations (75·1) are

$$\phi = c_1 \sin k_1 t + c_2 \cos k_1 t + c_3 \sin k_2 t + c_4 \cos k_2 t, \tag{75·5}$$

$$\theta = c_2 \sin k_1 t - c_1 \cos k_1 t - c_4 \sin k_2 t + c_3 \cos k_2 t, \tag{75·6}$$

the same as in Art. 69. To find the constants of integration we first find $\dot{\phi}$ and $\dot{\theta}$ from (75·5) and (75·6) and then assume the same initial conditions as in the first case in Art. 69, namely,

$$\phi = 0, \quad \theta = 0, \quad \dot{\phi} = 0, \quad \dot{\theta} = \dot{\theta}_0, \quad \text{when } t = 0.$$

We thus find
$$\phi = \frac{\dot{\theta}_0}{k_1 + k_2}(\cos k_1 t - \cos k_2 t),$$

$$\theta = \frac{\dot{\theta}_0}{k_1 + k_2}(\sin k_1 t + \sin k_2 t).$$

On replacing $k_1 + k_2$ by its value from (75·2), we get

$$\phi = \frac{A\dot{\theta}_0}{\sqrt{\{C^2\omega^2 + 4A(Wl + C\omega\dot{\psi})\}}}(\cos k_1 t - \cos k_2 t), \tag{75·7}$$

$$\theta = \frac{A\dot{\theta}_0}{\sqrt{\{C^2\omega^2 + 4A(Wl + C\omega\dot{\psi})\}}}(\sin k_1 t + \sin k_2 t). \tag{75·8}$$

If the rotation in azimuth is *opposite* the direction of the rotor spin, the equations of motion are found from (75·7) and (75·8) by changing the sign of $\dot{\psi}$ and replacing k's by k''s. Hence for this case we have

$$\phi = \frac{A\dot{\theta}_0}{\sqrt{\{C^2\omega^2 + 4A(Wl - C\omega\dot{\psi})\}}}(\cos k_1' t - \cos k_2' t), \tag{75·9}$$

$$\theta = \frac{A\dot{\theta}_0}{\sqrt{\{C^2\omega^2 + 4A(Wl - C\omega\dot{\psi})\}}}(\sin k_1' t + \sin k_2' t). \tag{75·10}$$

On comparing equations (75·7)–(75·10) with the case of no rotation ($\dot{\psi}=0$), we see that rotation of the axes in the same direction as the spin of the gyro decreases the amplitude of a disturbed motion (makes the motion more stable), and that rotation in the opposite direction increases the amplitude of such motion (makes the motion less stable). These facts furnish another reason for making a turn in the same direction as the spin of the gyro.

CHAPTER VIII

The Gyroscope as a Stabilizer (Continued)

The Ship Stabilizer

A. SHIP ON A STRAIGHT COURSE

76. Schlick or brake-type stabilizer

The idea of using a gyroscope to reduce the rolling of a ship at sea is due to Otto Schlick of Hamburg, Germany, who first installed a large pendulous gyroscope on a ship and carried out experiments with it in 1903. The gyro was installed with its axis of spin normally vertical and with its supporting frame resting on horizontal trunnions athwart the ship (Fig. 58). A pendulous weight w was attached to the bottom of the frame in order to make the center of gravity of gyro and frame fall below the horizontal axis of support, thus making the gyro and frame gravitationally stable.

From our previous study of the behavior of a gyroscope we can readily see that a rolling motion of the ship about a longitudinal axis* will cause a precessional to-and-fro motion of the gyro and its frame about the trunnion axis FF'. To limit the amplitude of such motion in the longitudinal plane, Schlick installed a braking mechanism B which applied a constant pressure on the axle. Although the brake type of stabilizer has been superseded by the active type invented by Elmer A. Sperry, the mathematical theory of the former type is instructive and will therefore be given here.

* The rolling of a ship is a to-and-fro rotation about a longitudinal horizontal axis. If the water offered no resistance to rolling, the axis of roll would pass through the center of gravity of the ship. But when the water offers resistance to rolling, as in fact it always does, the axis of roll passes through what is called the 'tranquil point', which is *above* the center of gravity. The rolling ship thus behaves somewhat as a pendulum swinging about a longitudinal axis through the tranquil point. See Peabody's *Naval Architecture* (1917), pp. 344–5.

When a ship floats in water, it displaces its own weight of water. The form of the displaced water changes as the ship rolls from side to side about a longitudinal axis. The center of gravity of the

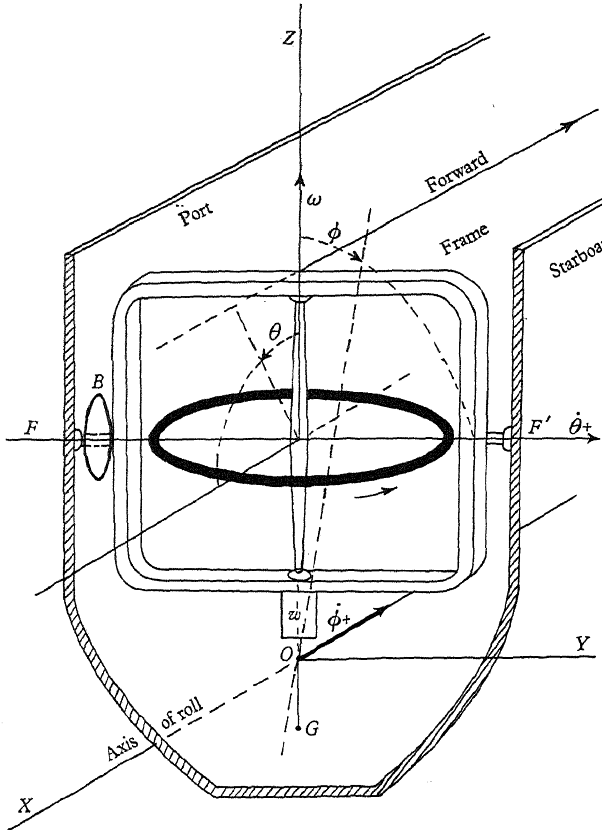


Fig. 58

displaced water is called the *center of buoyancy* and changes its position as the ship changes its inclination from the vertical.

Figure 59 represents a transverse section of a ship inclined from the vertical by an angle ϕ about a longitudinal axis. Here G denotes the center of gravity of the ship and B denotes the center of buoyancy. When a ship is floating freely in equilibrium and inclined at a small angle from the vertical, a vertical line through B intersects the stable vertical line at a point M, called the *metacenter*.

The distance $GM(=h)$ is called the *metacentric height.* It is evident that the weight of the ship and the buoyant force of the displaced water form a couple which tends to restore the ship to an upright position when M is above G. (If M were below G, the ship would capsize.)

In the mathematical treatment of the ship stabilizer we must take account of friction in two places: (1) the skin and obstructional friction between water and ship's hull, and (2) the brake friction on the precession axle of the gyroscope. We must also include the pulsating action of the waves on the ship.

The resisting moment of the skin and obstructional friction on the hull is proportional to the angular velocity of roll of the ship, and the moment of the brake friction on the precession axle is proportional to the angular velocity of precession. Hence with reference to Figs. 58 and 59, let:

$K\dot{\phi}=$ moment of skin and obstructional friction on hull,

$k\dot{\theta}=$ moment of brake friction on precession axle,

$M\sin bt=$ moment of wave action about axis of roll,

$I=$ moment of inertia of ship about axis of roll,

$A=$ moment of inertia of gyro and frame about trunnion axis,

$W=$ weight of ship,

$w=$ pendulous weight attached to gyro frame,

$h=$ metracentric height of ship,

$l=$ moment arm of w.

Then

$$Wh\sin\phi=Wh\phi=\text{righting moment about axis of roll},$$

$$wl\sin\theta=wl\theta=\text{moment of } w \text{ about trunnion axis}.$$

Now since the axis of roll is in the direction of the x-axis and the trunnion axis is in the direction of the y-axis (see Figs. 55 and 58), we have

$$\omega_{x'}=-\dot{\phi}\cos\theta,\quad \omega_{y'}=\dot{\theta},\quad \omega_{z'}=-\dot{\phi}\sin\theta.$$

Hence

$$H_{x'}=-I\dot{\phi}\cos\theta,\quad H_{y'}=A\dot{\theta},\quad H_{z'}=C(\omega-\dot{\phi}\sin\theta)=C\omega,$$

since $\dot{\phi}$ and θ are assumed to be small. Then

$$\frac{dH_{x'}}{dt} = -I(\ddot{\phi}\cos\theta - \dot{\phi}\dot{\theta}\sin\theta) = -I\ddot{\phi}\cos\theta,$$

$$\frac{dH_{y'}}{dt} = A\ddot{\theta}, \quad \frac{dH_{z'}}{dt} = 0.$$

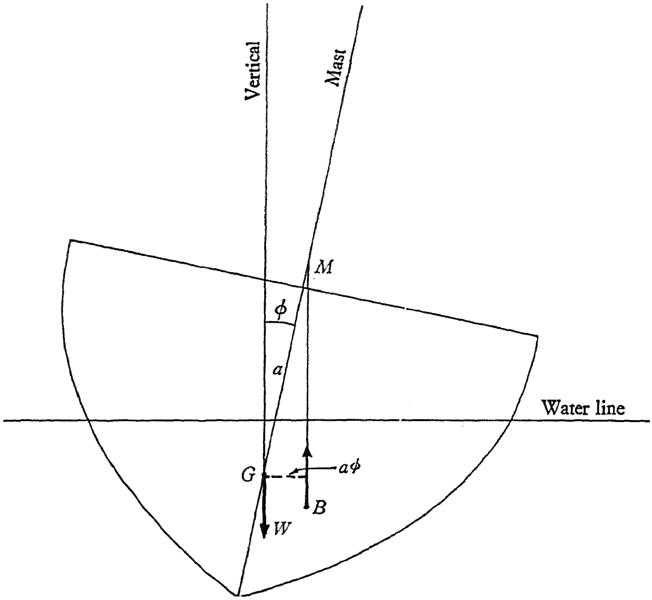


Fig. 59

Now bearing in mind the directions of the several moments and substituting the above quantities into equations (16·4), we have

$$-I\ddot{\phi}\cos\theta + C\omega\dot{\theta} = Wh\phi + K\dot{\phi} - M\sin bt,$$

$$A\ddot{\theta} + C\omega\dot{\phi}\cos\theta = -wl\theta - k\dot{\theta},$$

or, since $\cos\theta = 1$ approximately,

$$\left.\begin{array}{ll} (a) & I\ddot{\phi} + K\dot{\phi} - C\omega\dot{\theta} + Wh\phi = M\sin bt, \\ (b) & A\ddot{\theta} + k\dot{\theta} + C\omega\dot{\phi} + wl\theta = 0. \end{array}\right\} \qquad (76\cdot 1)$$

These are the differential equations of motion of the ship about its axis of roll and of the gyroscope about the trunnion axis.

To eliminate ϕ we first differentiate (76·1 (a)) and get

$$(c)\quad I\dddot{\phi}+K\ddot{\phi}-C\omega\ddot{\theta}+Wh\dot{\phi}=Mb\cos bt.$$

Then from (76·1 (b)) we get:

$$\dot{\phi}=-\frac{1}{C\omega}(A\ddot{\theta}+k\dot{\theta}+wl\theta),$$

$$\ddot{\phi}=-\frac{1}{C\omega}(A\dddot{\theta}+k\ddot{\theta}+wl\dot{\theta}),$$

$$\dddot{\phi}=-\frac{1}{C\omega}(A\theta^{\text{iv}}+k\dddot{\theta}+wl\ddot{\theta}).$$

Substituting these into (c), we have

$$AI\theta^{\text{iv}}+(Ik+AK)\dddot{\theta}+(Iwl+Kk+C^2\omega^2+Wah)\ddot{\theta}$$
$$+(Kwl+Whk)\dot{\theta}+Whwl\theta=-C\omega Mb\cos bt. \quad (76\cdot2)$$

Here the auxiliary equation for the left member equated to zero is

$$AIr^4+(Ik+AK)r^3+(Iwl+Kk+C^2\omega^2+WAh)r^2$$
$$+(Kwl+Whk)r+Whwl=0. \quad (76\cdot3)$$

To eliminate θ and get a fourth-order equation in ϕ, we proceed in a similar manner by first differentiating (76·1 (b)) and then substituting into the resulting equation the values of $\dot{\theta}$, $\ddot{\theta}$ and $\dddot{\theta}$ obtained from (76·1 (a)). The result is

$$AI\phi^{\text{iv}}+(AK+Ik)\dddot{\phi}+(AWh+Kk+C^2\omega^2+Iwl)\ddot{\phi}$$
$$+(Whk+Kwl)\dot{\phi}+Whwl\phi=M(wl-Ab^2)\sin bt+Mbk\cos bt.$$
$$(76\cdot4)$$

The auxiliary equation for (76·4) is seen to be the same as (76·3) above, as it should be.

In order to study the motion completely in a numerical case, it would be necessary to find all the roots of the auxiliary equation corresponding to (76·3).

The characteristic equation for the system (76·1) can also be found by a shorter and more general method. Let us put

$$\phi=ae^{rt}, \qquad \theta=be^{rt}.$$

Then

$$\dot{\phi}=are^{rt}, \quad \dot{\theta}=bre^{rt},$$

$$\ddot{\phi}=ar^2e^{rt}, \quad \ddot{\theta}=br^2e^{rt}.$$

Substituting these into the system

$$I\ddot{\phi}+K\dot{\phi}-C\omega\dot{\theta}+Wh\phi=0,$$

$$A\ddot{\theta}+k\dot{\theta}+C\omega\dot{\phi}+wl\theta=0,$$

and dividing the results throughout by e^{rt}, we get

$$(Ir^2+Kr+Wh)\,a-C\omega rb=0,$$

$$C\omega ra+(Ar^2+kr+wl)\,b=0.$$

These homogeneous equations in a and b will have a common solution if, and only if, the determinant of their coefficients is equal to zero, or

$$\begin{vmatrix} Ir^2+Kr+Wh & -C\omega r \\ C\omega r & Ar^2+kr+wl \end{vmatrix}=0.$$

Expanding this determinant and arranging the terms in descending powers of r, we get (76·3).

In order that the motion be stable, the values of ϕ and θ must not increase indefinitely with time. This means that the roots of (76·3) must either be pure imaginaries or that the real parts of the complex roots must be negative. A necessary condition for stable motion is that all the terms in (76·3) must be positive.* Then since $Whwl$ must be positive, h and l must have the same sign. This in turn means that the weight w must be placed *below* the trunnion axis (see footnote, Art. 76).

Continuing with the solution of (76·2), we find a particular solution by putting

$$\theta=R\sin bt+S\cos bt, \tag{76·5}$$

where R and S are constants to be determined. From (76·5) we get

$$\dot{\theta}=Rb\cos bt-Sb\sin bt,$$

$$\ddot{\theta}=-Rb^2\sin bt-Sb^2\cos bt,$$

$$\dddot{\theta}=-Rb^3\cos bt+Sb^3\sin bt,$$

$$\theta^{\text{iv}}=Rb^4\sin bt+Sb^4\cos bt.$$

* See Routh's *Stability of Motion*, p. 14, or *Advanced Rigid Dynamics* (6th edition, 1905), pp. 219–23.

Substituting into (76·2) these values of θ, $\dot{\theta}$, etc., and then equating the coefficients of $\sin bt$ and $\cos bt$ on the two sides of the equation, we get

$$\left.\begin{aligned}&[AIb^4-(Iwl+Kk+C^2\omega^2+WAh)\,b^2+Whwl]\,R\\&\qquad\qquad+[(Ik+AK)\,b^3-b(Kwl+kWh)]\,S=0,\\&[(Kwl+kWh)\,b-(Ik+AK)\,b^3]\,R\\&\qquad+[AIb^4-(Iwl+Kk+C^2\omega^2+WAh)\,b^2+Whwl]\,S=-C\omega bM.\end{aligned}\right\} \tag{76·6}$$

For brevity, let us put

$$E=AIb^4-(Iwl+Kk+C^2\omega^2+WAh)\,b^2+Whwl,$$
$$F=(Ik+AK)\,b^3-(Kwl+kWh)\,b.$$

Then equations (76·6) become

$$ER+FS=0,$$
$$FR-ES=C\omega bM.$$

Solving these for R and S, we find

$$R=\frac{F}{E^2+F^2}(C\omega bM),\quad S=-\frac{E}{E^2+F^2}(C\omega bM).$$

Then (76·5) becomes

$$\theta=\frac{C\omega bM}{E^2+F^2}(F\sin bt-E\cos bt),$$

which can be written in the form

$$\theta=\frac{C\omega bM}{\sqrt{(E^2+F^2)}}\sin(bt-\alpha), \tag{76·7}$$

where $$\alpha=\tan^{-1}\frac{E}{F}.$$

To find a particular solution of (76·4) we put

$$\phi=L\sin bt+N\cos bt, \tag{76·8}$$

where L and N are constants to be determined. Then

$$\dot{\phi}=Lb\cos bt-Nb\sin bt,$$
$$\ddot{\phi}=-Lb^2\sin bt-Nb^2\cos bt,$$
$$\dddot{\phi}=-Lb^3\cos bt+Nb^3\sin bt,$$
$$\phi^{\mathrm{iv}}=Lb^4\sin bt+Nb^4\cos bt.$$

Substituting into (76·4) these values of ϕ, $\dot{\phi}$, etc., and equating coefficients of $\sin bt$ and $\cos bt$ on the two sides of the equation, we get

$$\left.\begin{aligned} &[AIb^4-(AWh+Kk+C^2\omega^2+Iwl)\,b^2+Whwl]\,L \\ &\quad+[(AK+Ik)\,b^3-(Whk+Kwl)\,b]\,N=M(wl-Ab^2), \\ &\quad-[(AK+Ik)\,b^3-(Whk+Kwl)\,b]\,L \\ &\quad+[AIb^4-(AWh+Kk+C^2\omega^2+Iwl)\,b^2+Whwl]\,N=Mbk. \end{aligned}\right\} \quad (76\cdot9)$$

Replacing the bracketed quantities by E and F as before, we have

$$EL+FN=M(wl-Ab^2),$$

$$-FL+EN=Mbk.$$

Solving these for L and N, we get

$$L=\frac{M(wl-Ab^2)\,E-(Mbk)\,F}{E^2+F^2},$$

$$N=\frac{M(wl-Ab^2)\,F+(Mbk)\,E}{E^2+F^2}.$$

Then (76·8) becomes

$$\phi=\frac{M}{E^2+F^2}\{[(wl-Ab^2)\,E-bkF]\sin bt+[(wl-Ab^2)\,F+bkE]\cos bt\},$$

which can be written in the form

$$\phi=\frac{M}{\sqrt{(E^2+F^2)}}\sqrt{\{(wl-Ab^2)^2+b^2k^2\}}\sin(bt+\beta), \qquad (76\cdot10)$$

where $$\beta=\tan^{-1}\frac{(wl-Ab^2)\,F+bkE}{(wl-Ab^2)\,E-bkF}.$$

Equations (76·7) and (76·10) give the forced oscillations due to the pulsating action of the waves on the ship.

77. Roll of the ship with gyro clamped

For purposes of comparison we now investigate the rolling of the ship when the gyro is clamped. On putting $\omega=0$ in (76·1), we get, since in this case $\theta=\dot{\theta}=\ddot{\theta}=0$,

$$I\ddot{\phi}+K\dot{\phi}+Wh\phi=M\sin bt. \qquad (77\cdot1)$$

Here the auxiliary equation is

$$Ir^2+Kr+Wh=0.$$

$$\therefore \quad r=\frac{-K\pm\sqrt{(K^2-4IWh)}}{2I}=\frac{-K}{2I}\pm\frac{i\sqrt{(4IWh-K^2)}}{2I},$$

if $K^2<4IWh$.

Hence

$$\phi=e^{-Kt/2I}\left[C_1\cos\left(t\frac{\sqrt{(4IWh-K^2)}}{2I}\right)+C_2\sin\left(t\frac{\sqrt{(4IWh-K^2)}}{2I}\right)\right]. \tag{77·2}$$

Here the natural period of vibration is

$$T=2\pi\frac{2I}{\sqrt{(4IWh-K^2)}}=\frac{4\pi I}{\sqrt{(4IWh-K^2)}}. \tag{77·3}$$

To find a particular solution of (77·1) we put

$$\phi=C_3\sin bt+C_4\cos bt. \tag{77·4}$$

Then
$$\dot\phi=C_3b\cos bt-C_4b\sin bt,$$
$$\ddot\phi=-C_3b^2\sin bt-C_4b^2\cos bt.$$

Substituting these into (77·1) and then equating coefficients of $\sin bt$ and $\cos bt$ on the two sides of the equation, we get

$$(Wh-Ib^2)C_3-KbC_4=M,$$
$$KbC_3+(Wh-Ib^2)C_4=0.$$

Solving these for C_3 and C_4, we find

$$C_3=\frac{M(Wh-Ib^2)}{(Wh-Ib^2)^2+K^2b^2},$$

$$C_4=\frac{-MKb}{(Wh-Ib^2)^2+K^2b^2}.$$

Hence (77·4) becomes

$$\phi=\frac{M}{(Wh-Ib^2)^2+K^2b^2}[(Wh-Ib^2)\sin bt-Kb\cos bt],$$

which reduces to

$$\phi=\frac{M}{\sqrt{\{(Wh-Ib^2)^2+K^2b^2\}}}\sin(bt-\gamma), \tag{77·5}$$

where
$$\gamma=\tan^{-1}\frac{Kb}{Wh-Ib^2}.$$

Equation (77·5) gives the forced oscillation of the ship due to the action of the waves.

78. Numerical example

The following data, according to Perry,* are approximately correct for a ship of 6000 tons displacement, having a metacentric height of 1.5 feet, a natural roll period of 14 seconds, and equipped with a gyro 12 feet in diameter, weighing 10 tons, and spinning at 100 radians per second, all data being in foot-pound-second units:

$$I = 10^8, \quad A = 7\times 10^3, \quad C\omega = 2.5\times 10^6, \quad K = 4\times 10^6,$$

$$k = 5\times 10^4, \quad Wh = 2\times 10^7, \quad wl = 7\times 10^4.$$

Substituting these data into (76·3) and then dividing through by 7×10^{11}, we get

$$r^4 + 7.18r^3 + 19.41r^2 + 1.83r + 2 = 0. \tag{78·1}$$

Our first problem is to determine the nature of the roots of (78·1) and then find them. We could proceed to find the roots directly by Graeffe's root-squaring method, without previously determining their nature, but in dynamical problems it is better to determine the nature of the roots before trying to find them.

If the characteristic or auxiliary equation of a fourth-order differential equation be written in the form

$$x^4 + px^3 + qx^2 + rx + s = 0, \tag{78·2}$$

the necessary and sufficient conditions that the motion determined by the given differential equation shall be stable (that is, that the real parts of the complex roots of (78·2) shall be negative or zero) is that all terms in (78·2) must be positive and that

$$pqr - r^2 - p^2s \geqslant 0.\dagger$$

These conditions are fulfilled in equation (78·1), for

$$(7.18)(19.41)(1.83) - (1.83)^2 - (7.18)^2(2) = 148.6.$$

Furthermore, if a quartic equation be written in the form

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0, \tag{78·3}$$

let $H = ac - b^2$,

$I = ae - 4bd + 3c^2$,

$J = ace + 2bcd - ad^2 - eb^2 - c^3$,

$\Delta = I^3 - 27J^2$.

* 'The Use of Gyrostats', *Nature, Lond.*, 12 March 1908.

† Routh, *Stability of Motion*, p. 14, or *Advanced Rigid Dynamics* (6th edition, 1905), pp. 219–23.

Then the conditions that all the roots of (78·3) be complex are

$$\Delta > 0,$$

and $$H \text{ and } a^2I - 12H^2 \text{ not both negative.*}$$

Applying these conditions to equation (78·1), where $a=1$, $b=1.795$, $c=3.24$, $d=0.46$, $e=2$, we find

$$H=0.02, \quad I=32.67, \quad J=-28.83,$$

$$\Delta=12{,}430, \quad a^2I-12H^2=32.67.$$

Hence all the conditions for complex roots are satisfied, and we now know that all the roots of (78·1) are complex and that the real parts of the roots are negative.

In order to find the roots of (78·1) we shall not apply Graeffe's method but shall find them with less labor by utilizing all our knowledge concerning them. Since the roots are all complex and their real parts are negative, we represent them by

$$-\alpha_1 \pm i\beta_1 \quad \text{and} \quad -\alpha_2 \pm i\beta_2,$$

or $$-\alpha_1+i\beta_1, \quad -\alpha_1-i\beta_1, \quad -\alpha_2+i\beta_2, \quad -\alpha_2-i\beta_2.$$

Then from the well-known relations between the roots and coefficients in a rational integral equation,† we have

$$\left.\begin{array}{lr} (a) & \alpha_1+\alpha_2=3.59, \\ (b) & \alpha_1^2+\beta_1^2+\alpha_2^2+\beta_2^2+4\alpha_1\alpha_2=19.41, \\ (c) & \alpha_2(\alpha_1^2+\beta_1^2)+\alpha_1(\alpha_2^2+\beta_2^2)=0.915, \\ (d) & (\alpha_1^2+\beta_1^2)(\alpha_2^2+\beta_2^2)=2. \end{array}\right\} \qquad (78\cdot4)$$

We now utilize some additional knowledge concerning the roots of equation (78·1). We know from our previous study of the

* Burnside and Panton, *Theory of Equations*, vol. 1 (1899), p. 145.

§ If a_1, a_2, a_3, a_4 are the roots of the quartic equation

$$x^4+ax^3+bx^2+cx+d=0,$$

these relations are

$$\begin{aligned} a&=-(a_1+a_2+a_3+a_4), \\ b&=a_1a_2+a_1a_3+a_1a_4+a_2a_3+a_2a_4+a_3a_4, \\ c&=-(a_1a_2a_3+a_1a_2a_4+a_1a_3a_4+a_2a_3a_4), \\ d&=a_1a_2a_3a_4. \end{aligned}$$

gyroscopic pendulum that β_1, which corresponds to k_1 of Art. 69, is very small; and since the slow vibrations are not quickly damped out, we may infer that α_1 is also small. We also know that β_2, which corresponds to k_2 of Art. 69, is fairly large; and we may also suppose that α_2 is also large, since the rapid oscillations are soon damped out. We can therefore get approximate values for the smaller roots of equations (78·1) by neglecting the first two terms, and we can get approximate values for the larger roots by neglecting the last two terms.

Hence to find the approximate values of the smaller roots we have

$$19.41r^2+1.83r+2=0.$$

$$\therefore \quad r=\frac{-1.83\pm\sqrt{\{(1.83)^2-8(19.41)\}}}{38.82}=-0.047\pm 0.317i.$$

Neglecting the last two terms of (78·1) to find approximate values of the larger roots, we have

$$r^2+7.18r+19.41=0.$$

$$\therefore \quad r=-3.59\pm\sqrt{\{(3.59)^2-19.41\}}=-3.59\pm 2.55i.$$

Approximate values for α_1, α_2, β_1, β_2 are therefore

$$\alpha_1=0.047, \quad \alpha_2=3.59, \quad \beta_1=0.317, \quad \beta_2=2.55.$$

We now proceed to find more accurate values by the method of iteration. Starting with $\alpha_1=0.047$, we substitute this into (78·4 (a)) and get

$$\alpha_2=3.59-0.047=3.543.$$

Then substituting the value of $4\alpha_1\alpha_2=4(0.047)(3.543)=0.67$ into 78·4 (b), we have

$$(\alpha_1^2+\beta_1^2)+(\alpha_2^2+\beta_2^2)=19.41-0.67=18.74.$$

For brevity, put $\alpha_1^2+\beta_1^2=x$, $\alpha_2^2+\beta_2^2=y$. Then

$$x+y=18.74$$

and

$$xy=2,$$

from (78·4 (d)). From these we get $x=0.11$, $y=18.18$. That is,

$$\alpha_1^2+\beta_1^2=0.11,$$

$$\alpha_2^2+\beta_2^2=18.18.$$

Substituting these and α_2 into (78·4 (c)), we have

$$3{\cdot}543\,(0.11)+\alpha_1(18.18)=0.915,$$

whence $$\alpha_1=0.0289.$$

We now repeat the iteration by substituting this value of α_1 into (78·4 (a)), thus getting $\alpha_2=3.56$. Then

$$4\alpha_1\alpha_2=4(0.0289)\,(3.56)=0.41.$$

Substituting this into (78·4 (b)), we have

$$(\alpha_1^2+\beta_1^2)+(\alpha_2^2+\beta_2^2)=19.41-0.41=19.00.$$

Then $$x+y=19,$$

$$xy=2,$$

from which $x=0.106$, $y=18.87$. That is,

$$\alpha_1^2+\beta_1^2=0.106, \quad \alpha_2^2+\beta_2^2=18.87.$$

Now substituting these into (78·4 (c)), we have

$$\alpha_1(18.87)+3{\cdot}56(0.106)=0.915,$$

whence $$\alpha_1=0.0285.$$

We now iterate once more by substituting this value of α_1 into (78·4 (a)), giving $\alpha_2=3.56$. Then $4\alpha_1\alpha_2=0.406$, and (78·4(b)) becomes

$$(\alpha_1^2+\beta_1^2)+(\alpha_2^2+\beta_2^2)=19.41-0.406=19.004.$$

Hence we have $$x+y=19.004,$$

$$xy=2,$$

from which $x=0.106$, $y=18.87$. Then substituting these and α_2 into (78·4 (c)), we get $\alpha_1=0.0285$ as before. Since these values can not be improved by further iteration, we take them as correct.

From $\alpha_1^2+\beta_1^2=0.106$ and $\alpha_1=0.0285$, we get

$$\beta_1^2=0.106-(0{\cdot}0285)^2=0.1052,$$

$$\beta_1=0.324.$$

Also, $\alpha_2^2+\beta_2^2=18.87$ and $\alpha_2=3.56$. Hence

$$\beta_2^2=18.87-(3{\cdot}56)^2=6\ 1964,$$

$$\beta_2=2.49.$$

The equation for the natural vibration of the ship with gyro spinning is therefore

$$\phi = e^{-0.0285t}(A\cos 0.324t + B\sin 0.324t)$$
$$+ e^{-3\cdot 56t}(C\cos 2.49t + D\sin 2.49t), \qquad (78\cdot 5)$$

where A, B, C, D are constants of integration to be determined from initial conditions.

The periods of oscillation are

$$T_1 = \frac{2\pi}{0.324} = 19.4 \text{ seconds}, \quad T_2 = \frac{2\pi}{2.49} = 2.52 \text{ seconds}.$$

For purposes of comparison to determine the effect of the gyro on the natural oscillation period of the ship, we compute the oscillation period when the gyro is clamped. On substituting into equation (77·2) the numerical values of I, K and Wh, we find

$$\phi = e^{-0.02t}(C_1\cos 0.447t + C_2\sin 0.447t). \qquad (78\cdot 6)$$

Here the period is

$$T = \frac{2\pi}{0.447} = 14.06 \text{ seconds}.$$

The gyro stabilizer thus lengthened the natural oscillation period by 37.7 % and also caused the amplitude of the oscillations to decrease more rapidly.

We have yet to examine the forced oscillations. From equation (76·7) we found the amplitude of θ due to the pulsating force of the waves to be

$$(\text{Amp.})_\theta = \frac{C\omega bM}{\sqrt{(E^2+F^2)}}.$$

In order to indicate the effectiveness of the stabilizer in preventing the rolling of a ship in a rough sea, we shall consider the case of a ship rolling among high waves having a period near the natural (free) oscillation period of the ship. In this case the effect of the waves will be cumulative, thereby causing the ship to roll with wide amplitude. We take $b = 0.455$, thus giving a period of

$$T = \frac{2\pi}{0.455} = 13.8 \text{ seconds}$$

for the forced oscillations of the ship.

On substituting the numerical values of the quantities occurring in E and F as given in Art. 76, we find

$$\sqrt{(E^2+F^2)}=141.4\times 10^{10}.$$

Then from equation (76·7) we have

$$(\text{Amp.})_\theta=\frac{M}{141.4\times 10^{10}}(2.5\times 10^6\times 0.455)$$

$$=\frac{113.75M}{141.4\times 10^6}.$$

From equation (76·10) we have

$$(\text{Amp.})_\phi=\frac{M}{\sqrt{(E^2+F^2)}}\sqrt{\{(wl-Ab^2)^2+b^2k^2\}}$$

$$=\frac{7.223M}{141.4\times 10^6}=\frac{0.0511M}{10^6}.$$

Here we notice in passing that

$$\frac{(\text{Amp.})_\theta}{(\text{Amp.})_\phi}=\frac{113.75}{7.223}=15.75,$$

or

$$(\text{Amp.})_\theta=15.75\ (\text{Amp.})_\phi.$$

The amplitude of the forced oscillations when the gyro is clamped is given by equation (77·5). Hence we have

$$(\text{Amp.})_\phi=\frac{M}{\sqrt{\{(Wh-Ib^2)^2+K^2b^2\}}}$$

$$=\frac{M}{1.950\times 10^6}=\frac{0.513M}{10^6},$$

which is ten times as great as the amplitude when the stabilizer was acting.

The foregoing study shows two marked effects of a gyroscopic stabilizer on the rolling of ships:

(1) The stabilizer reduces the angular velocity of rolling and thereby increases the period of rolling, thus making the period of rolling greater (in most cases) than the period of the waves.* This

* One of the longest periods of waves on record was 16·5 seconds, and this occurred with waves 46 feet high. See Gaillard, D.D., *Wave Action in Relation to Engineering Structures*, Washington, 1904.

increases in the period of rolling prevents in large measure the cumulative effects of the oncoming waves.

(2) The stabilizer makes a drastic decrease in the amplitude of rolling, especially when the period of the waves is nearly the same as the natural oscillation period of the ship.

Fig. 60 shows graphically the marked effect of the stabilizer on the amplitude of rolling.

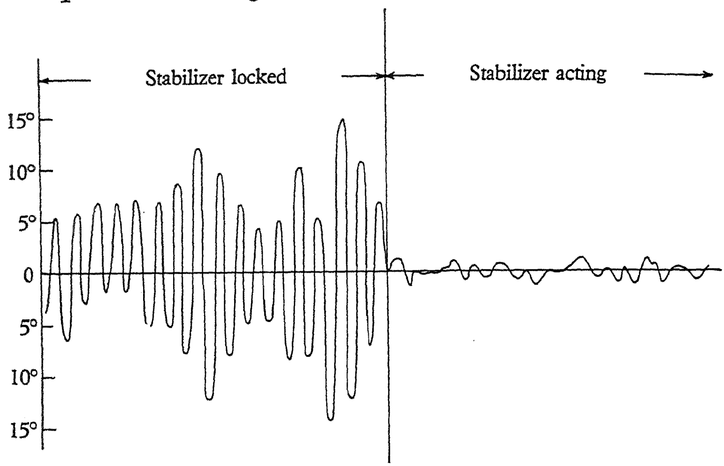


Fig. 60

79. Sperry or active-type stabilizer

Water waves do not beat against a floating ship as they do against a solid sea wall, because of the fact that a ship is free to rise and fall and thereby permit the bulk of a wave to pass under it. Hence any single wave imparts only a small amount of roll to the ship. But when each of a continuous train of waves imparts a small increment of roll, a large roll is soon built up if the period of the waves happens to be nearly the same as the natural period of roll of the ship.

The idea of the Sperry stabilizer is to quench each increment of roll as soon as it occurs, thus preventing the building up of a large roll. This is accomplished by causing a forced precession of the gyro about the trunnion axis in the proper direction to oppose the roll.

The Sperry stabilizer consists of a large gyro with spin axis normally vertical, a small control gyro with spin axis horizontal and athwart the ship, a precession motor, and a magnetic brake, plus

the motors required for spinning the gyros. The casing of the control gyro is free to precess about a vertical axis. Horizontal pins p_1 and p_2 project from the casing of the control gyro, as indicated in Fig. 61. Centralizing springs are attached to pin p_2. Pin p_1 is free to swing back and forth between electric contacts c_1 and c_2.

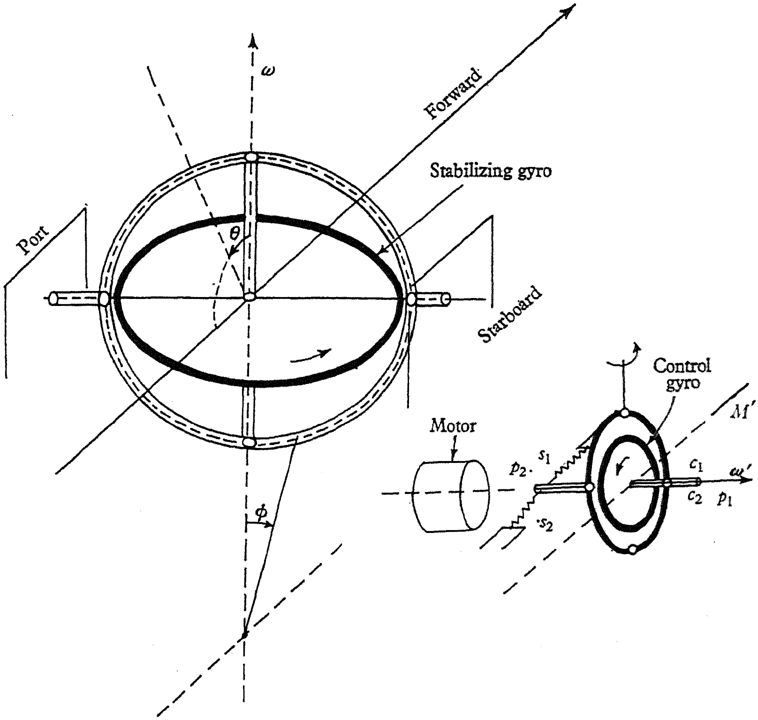


Fig. 61

The operation of the Sperry stabilizer is as follows: When the ship rolls to starboard, for example, by as much as a single degree, the control gyro precesses about a vertical axis as indicated in the figure. This precession causes pin p_1 to make electrical contact at c_1, thereby starting the precession motor and causing it to exert a torque about the trunnion axis in a clockwise direction as viewed from the starboard side. When the ship rolls back in the opposite direction, the casing of the control gyro precesses about the vertical axis in the opposite direction, causing pin p_1 to make

electrical contact at c_2 soon after the ship has rolled past its vertical position. This starts the precession motor in the opposite direction, causing it to precess the main gyro in the reversed direction so as to oppose the roll again.

The Sperry stabilizer also retards the rolling in other ways. If the rolling of the ship is fast enough to overdrive the motor, it thereby converts the motor into a generator for the time being. Since a part of the kinetic energy of rolling is expended in driving the generator, the rolling is reduced to some extent.

Furthermore, as the current taken by the motor diminishes, the magnetic brake comes into action and reduces the roll.

In order to obtain the differential equations of motion of the ship under the action of the Sperry stabilizer, let us suppose that the main or stabilizing gyro is forced to precess about the trunnion axis in a clockwise direction as viewed from the starboard side (see Fig. 61). Then since the precession vector is directed toward the port side, the rule for the direction of the gyroscopic reaction moment shows that this moment tends to roll the ship toward the port side. Hence if the ship begins to roll to starboard, this rolling can be counteracted by a forced precession of the main gyro in the direction of negative θ. It is to be noted that this is also the direction of the G.R.M. induced by the rolling.

Since the main gyro and the control gyro are both nonpendulous in the Sperry stabilizer (pendulosity is not needed), and since no brake friction moment is constantly applied about the trunnion axis, we can obtain the differential equations for the Sperry stabilizer by putting $w=0$ and $k=0$ in equations (76·1) and writing in a term for the forced precession moment about the trunnion axis. Hence the differential equations of motion are

$$\left.\begin{aligned} I\ddot{\phi}+K\dot{\phi}-C\omega\dot{\theta}+Wh\phi &= M\sin bt, \\ A\ddot{\theta}+C\omega\dot{\phi} &= -N, \end{aligned}\right\} \qquad (79\cdot 1)$$

where N denotes the variable moment about the trunnion axis.

Because of the slight gaps with respect to time in the train of operations of the Sperry stabilizer, the rolling of a ship under the action of the stabilizer probably cannot be represented exactly by a continuous relation between ϕ and t. However, since the action of the stabilizer is controlled by the rolling of the ship, we may reasonably assume that the applied precessional moment about the

trunnion axis is proportional to the angular velocity of roll of the ship. The applied precessional moment will then supplement the gyroscopic moment opposing the ship's rolling. The value of N in this case would be $Q\dot{\phi}$, where Q denotes a positive constant.

This assumed relationship leads to the result that by making $C\omega$ sufficiently large the amplitude of rolling can be kept as small as desired while the ship is following a straight course or a course which curves in the same direction as the spin of the gyro; but the result also shows that the ship will be unstable if its course curves in the direction opposite to the direction of spin of the gyro. In view of the fact that ships are stable under the action of the Sperry stabilizer when the course curves in any direction, the assumption that $N = Q\dot{\phi}$ is not valid.

Since a roll in the direction of positive ϕ can be quenched by a forced precession of the gyro spin axis in the direction of negative θ, we may reasonably assume that $\dot{\theta} = -m\dot{\phi}$, where m denotes a positive constant. This assumption leads to the result that the ship will be stable when its course curves in any direction. In this case the precessional moment about the trunnion axis is assumed to be a function of t and will be denoted by $R(t)$. It can be found, if desired, after ϕ has been found.

With the assumptions stated above, equations (79·1) are replaced by the following:

$$\left.\begin{array}{lr} (a) & I\ddot{\phi} + K\dot{\phi} - C\omega\dot{\theta} + Wh\phi = M\sin bt, \\ (b) & A\ddot{\theta} + C\omega\dot{\phi} = R(t), \\ (c) & \dot{\theta} = -m\dot{\phi}. \end{array}\right\} \qquad (79\cdot2)$$

On substituting (c) into (a), we get

$$I\ddot{\phi} + (K + C\omega m)\,\dot{\phi} + Wh\phi = M\sin bt. \qquad (79\cdot3)$$

Here the characteristic equation is

$$Ir^2 + (K + C\omega m)\,r + Wh = 0,$$

whence
$$r = \frac{-(K + C\omega m)}{2I} \pm \frac{\sqrt{\{(K + C\omega m)^2 - 4IWh\}}}{2I}.$$

If $(K + C\omega m)^2 \geqslant 4IWh$, both values of r are real and negative, and there is no natural oscillation. The motion is overdamped.

If $(K+C\omega m)^2 < 4IWh$, r becomes

$$r = \frac{-(K+C\omega m)}{2I} \pm i\frac{\sqrt{\{4IWh-(K+C\omega m)^2\}}}{2I}.$$

The free oscillations are then represented by the equation

$$\phi = \exp\left[-\frac{(K+C\omega m)t}{2I}\right](c_1 \sin\lambda t + c_2\cos\lambda t),$$

where
$$\lambda = \frac{\sqrt{\{4IWh-(K+C\omega m)^2\}}}{2I}.$$

Note that the presence of $C\omega$ in the exponent of e makes the damping much more rapid.

To find a particular solution of (79·3), we put

$$\phi = P_1 \sin bt + P_2 \cos bt.$$

Then
$$\dot\phi = P_1 b\cos bt - P_2 b \sin bt,$$
$$\ddot\phi = -P_1 b^2 \sin bt - P_2 b^2 \cos bt.$$

Substituting these into (79·3) and equating the coefficients of $\sin bt$ and $\cos bt$ on the two sides of the equation, we get

$$(Wh-Ib^2)P_1 - b(K+C\omega m)P_2 = M,$$
$$b(K+C\omega m)P_1 + (Wh-Ib^2)P_2 = 0.$$

Solving for P_1 and P_2, we find

$$P_1 = \frac{M(Wh-Ib^2)}{(Wh-Ib^2)^2+b^2(K+C\omega m)^2},$$
$$P_2 = \frac{-Mb(K+C\omega m)}{(Wh-Ib^2)^2+b^2(K+C\omega m)^2}.$$

A particular solution is therefore

$$\phi = \frac{M}{(Wh-Ib^2)^2+b^2(K+C\omega m)^2} \times [(Wh-Ib^2)\sin bt - b(K+C\omega m)\cos bt],$$

which can be written as

$$\phi = \frac{M}{\sqrt{\{(Wh-Ib^2)^2+b^2(K+C\omega m)^2\}}}\sin(bt-\alpha), \qquad (79{\cdot}4)$$

where
$$\alpha = \tan^{-1}\frac{b(K+C\omega m)}{Wh-Ib^2}.$$

Here it is evident that the amplitude of ϕ is less than it would be if $\omega=0$, and that it can be decreased at pleasure by increasing $C\omega$ sufficiently. It is also to be noted that when $\omega=0$, equation (79·4) reduces to (77·5).

We can now find $R(t)$, if desired, by means of (b) and (c) of (79·2). From (79·4) we have

$$\dot{\phi}=\frac{bM}{\sqrt{\{(Wh-Ib^2)^2+b^2(K+C\omega m)^2\}}}\cos(bt-\alpha).$$

Then from (c),

$$\dot{\theta}=\frac{-mbM}{\sqrt{\{(Wh-Ib^2)^2+b^2(K+C\omega m)^2\}}}\cos(bt-\alpha).$$

Hence
$$\ddot{\theta}=\frac{mb^2M}{\sqrt{\{(Wh-Ib^2)^2+b^2(K+C\omega m)^2\}}}\sin(bt-\alpha).$$

Substituting into (79·2 (b)) these values of $\ddot{\theta}$ and $\dot{\phi}$, we get

$$R(t)=\frac{Mb}{\sqrt{\{(Wh-Ib^2)^2+b^2(K+C\omega m)^2\}}}\times[Amb\sin(bt-\alpha)+C\omega\cos(bt-\alpha)]$$

$$=Mb\left[\frac{(Amb)^2+(C\omega)^2}{(Wh-Ib^2)^2+b^2(K+C\omega m)^2}\right]^{\frac{1}{2}}\sin(bt-\alpha+\beta), \quad (79\cdot5)$$

where
$$\beta=\tan^{-1}\frac{C\omega}{Amb}.$$

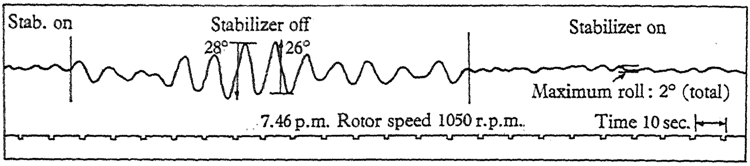
The precessional moment applied by the stabilizer is therefore periodic with the same period as ϕ but differing in phase by the angle β. Note that β is likely to be near 90°, because its tangent is directly proportional to $C\omega$.

Because of the fact that the Sperry stabilizer quenches the increments of roll as fast as they occur, a smaller gyro suffices for stabilizing a ship than is the case with the brake-type stabilizer. Throughout a cruise of 29,000 miles by a yacht of 3000 tons displacement, a Sperry stabilizer kept the amplitude of roll down to 1° (2° from one extreme to the other) by means of a rotor 8 feet in diameter, weighing 22 tons (long tons) and spinning at 1300 r.p.m.

An active-type stabilizer can not only quench the rolling of a moving ship, but it can also produce rolling of a ship at rest. By a change in the electrical connections at the switchboard, the stabilizer can be made to apply to the ship a periodic torque that will produce

rolling with increasing amplitude. If, for example, a ship equipped with such a stabilizer should run aground, the stabilizer might be able to loosen it from the ground and enable it to keep going.

Fig. 62 shows how effectively a Sperry stabilizer controlled the rolling of a small ship on one occasion.



RECORD TAKEN IN A SHIP OF 2000 TONS DISPLACEMENT

Unstabilized
Maximum roll, side to side, 28°
Average roll, 13°

Stabilized
Maximum roll, side to side, 2°
Average roll, 1°

Fig. 62

B. SHIP ON A CURVED PATH

80. Brake-type stabilizer

Let us assume that a ship equipped with a gyroscopic stabilizer is making a turn in a circular path. Let the ship have a constant angular velocity $\dot{\psi}$ and a constant linear velocity V. Then from the relations

$$V = R\dot{\psi} \quad \text{and} \quad a_n = \frac{V^2}{R},$$

where R denotes the radius of the circular path and a_n the centripetal acceleration of the ship, we find (on eliminating R)

$$a_n = V\dot{\psi}.$$

Then the magnitude of the centrifugal force acting at the center of gravity of the ship is

$$F = \frac{W}{g} a_n = \frac{WV\dot{\psi}}{g},$$

and the moment of this force about the longitudinal axis of roll is

$$M = \frac{pWV\dot{\psi}}{g},$$

where p denotes the distance from the center of gravity to the axis of roll. Since the axis of roll is above the c.g., this moment will tend to heel the ship inward toward the center of curvature.

Since the ship is moving on a curve, the axes of the gyro and frame are turning in azimuth with angular velocity $\dot{\psi}$. We shall assume that the gyro is spinning in a counterclockwise direction as viewed from above, and that the ship is moving in the same direction (swerving to the left) on the curve. Hence to derive the differential equations of motion we proceed as in Art. 75.

Referring again to Figs. 57 and 58, let the longitudinal axis of the ship lie in the direction of OX. Then the trunnion axis will lie in the direction of OY. Then as in Art. 75 the projections of $\dot{\psi}$ on the axes of the moving trihedron are

$$\begin{array}{lll} -\dot{\psi}\cos\phi\sin\theta & \text{on} & OX', \\ -\dot{\psi}\sin\phi & \text{on} & OY', \\ \dot{\psi}\cos\phi\cos\theta & \text{on} & OZ'. \end{array}$$

In the disturbed position of the gyro its angular velocities about the moving axes are

$$\begin{array}{lll} -\dot{\phi}\cos\theta & \text{about} & OX', \\ \dot{\theta} & \text{about} & OY', \\ -\dot{\phi}\sin\theta & \text{about} & OZ'. \end{array}$$

Hence the total angular velocities about the moving axes are

$$\begin{aligned} \omega_x &= -\dot{\phi}\cos\theta - \dot{\psi}\cos\phi\sin\theta, \\ \omega_y &= \dot{\theta} - \dot{\psi}\sin\phi, \\ \omega_z &= -\dot{\phi}\sin\theta + \dot{\psi}\cos\phi\cos\theta. \end{aligned}$$

The corresponding angular momenta are therefore

$$\begin{aligned} H_x &= -I(\dot{\phi}\cos\theta + \dot{\psi}\cos\phi\sin\theta), \\ H_y &= A(\dot{\theta} - \dot{\psi}\sin\phi), \\ H_z &= C(\omega - \dot{\phi}\sin\theta + \dot{\psi}\cos\phi\cos\theta). \end{aligned}$$

Hence

$$\frac{dH_x}{dt} = -I\ddot{\phi}, \quad \text{approx.},$$

$$\frac{dH_y}{dt} = A\ddot{\theta}, \quad \text{approx.},$$

$$\frac{dH_z}{dt} = 0, \quad \text{approx.},$$

after neglecting negligible products and replacing sines by angles and cosines by unity.

On substituting the above quantities into (16·4), we get

$$\left.\begin{aligned}(a)\quad & I\ddot{\phi}-C\omega\dot{\theta}+(C\omega\dot{\psi}+Wh)\phi+K\dot{\phi}=-\frac{pWV\dot{\psi}}{g}+M\sin bt,\\ (b)\quad & A\ddot{\theta}+C\omega\dot{\phi}+(C\omega\dot{\psi}+wl)\theta+k\dot{\theta}=0.\end{aligned}\right\} \tag{80·1}$$

Differentiating (*b*), we have

$$(c)\quad A\dddot{\theta}+C\omega\ddot{\phi}+(C\omega\dot{\psi}+wl)\dot{\theta}+k\ddot{\theta}=0.$$

Solving (*a*) for $\dot{\theta}$ and then differentiating the result twice, we have

$$\dot{\theta}=\frac{1}{C\omega}\left[I\ddot{\phi}+(C\omega\dot{\psi}+Wh)\phi+K\dot{\phi}+\frac{pWV\dot{\psi}}{g}-M\sin bt\right],$$

$$\ddot{\theta}=\frac{1}{C\omega}[I\dddot{\phi}+(C\omega\dot{\psi}+Wh)\dot{\phi}+K\ddot{\phi}-Mb\cos bt],$$

$$\dddot{\theta}=\frac{1}{C\omega}[I\phi^{\mathrm{iv}}+(C\omega\dot{\psi}+Wh)\ddot{\phi}+K\dddot{\phi}+Mb^2\sin bt].$$

Substituting these into (*c*) and rearranging slightly, we get

$$\begin{aligned}AI\phi^{\mathrm{iv}}+(AK+Ik)\dddot{\phi}+[C^2\omega^2+C\omega\dot{\psi}(A+I)+AWh+Iwl+Kk]\ddot{\phi}\\ +[C\omega\dot{\psi}(K+k)+Kwl+kWh]\dot{\phi}\\ +[C^2\omega^2\dot{\psi}^2+C\omega\dot{\psi}(Wh+wl)+Whwl]\phi\\ =-\frac{pWV\dot{\psi}}{g}(C\omega\dot{\psi}+wl)+(C\omega\dot{\psi}+wl)M\sin bt\\ +kbM\cos bt-AMb^2\sin bt.\end{aligned} \tag{80·2}$$

Here the characteristic equation is

$$\begin{aligned}AIr^4+(AK+Ik)r^3+[C^2\omega^2+C\omega\dot{\psi}(A+I)+AWh+Iwl+Kk]r^2\\ +[C\omega\dot{\psi}(K+k)+Kwl+kWh]r\\ +[C^2\omega^2\dot{\psi}^2+C\omega\dot{\psi}(Wh+wl)+Whwl]=0.\end{aligned} \tag{80·3}$$

Since all coefficients and the constant term are positive, the first condition for stability is fulfilled.

To find a particular solution of (80·2), we put

$$\phi=P+Q\sin bt+R\cos bt.$$

Then

$$\dot{\phi}=Qb\cos bt-Rb\sin bt,$$
$$\ddot{\phi}=-Qb^2\sin bt-Rb^2\cos bt,$$
$$\dddot{\phi}=-Qb^3\cos bt+Rb^3\sin bt,$$
$$\phi^{\mathrm{iv}}=Qb^4\sin bt+Rb^4\cos bt.$$

Substituting these into (80·2) and equating coefficients of like functions on the two sides of the equation, we get

$$P=\frac{-pWV\dot{\psi}(wl+C\omega\dot{\psi})}{g\,[C^2\omega^2\dot{\psi}^2+C\omega\dot{\psi}(Wh+wl)+Whwl]},$$

and

$$\left.\begin{aligned}EQ+FR&=M(C\omega\dot{\psi}+wl-Ab^2)=SM,\\-FQ+ER&=kbM,\end{aligned}\right\}\tag{80·4}$$

where

$$E=AIb^4-[C\omega\dot{\psi}(A+I)+C^2\omega^2+AWh+Iwl+Kk]\,b^2$$
$$+C^2\omega^2\dot{\psi}^2+C\omega\dot{\psi}(Wh+wl)+Whwl,$$
$$F=(AK+Ik)\,b^3-[C\omega\dot{\psi}(K+k)+Kwl+kWh]\,b,$$
$$S=C\omega\dot{\psi}+wl-Ab^2.$$

Solving equations (80·4) for Q and R, we find

$$Q=\frac{M(ES-kbF)}{E^2+F^2},\quad R=\frac{M(kbE+FS)}{E^2+F^2}.$$

A particular solution of (80·2) is therefore

$$\phi=\frac{-pWV\dot{\psi}(wl+C\omega\dot{\psi})}{g[C^2\omega^2\dot{\psi}^2+C\omega\dot{\psi}(Wh+wl)+Whwl]}+\frac{M}{E^2+F^2}$$
$$\times[(ES-kbF)\sin bt+(kbE+FS)\cos bt]$$
$$=\frac{-pWV\dot{\psi}(wl+C\omega\dot{\psi})}{g[C^2\omega^2\dot{\psi}^2+C\omega\dot{\psi}(Wh+wl)+Whwl]}$$
$$+M\sqrt{\left(\frac{S^2+k^2b^2}{E^2+F^2}\right)}\sin(bt+\beta),\tag{80·5}$$

where

$$\beta=\tan^{-1}\frac{kbE+FS}{ES-kbF}.$$

The negative sign of the constant term shows that the ship is inclined inward toward the center of curvature of the path, and the

result further shows that the ship oscillates about the inclined position.

If the path of the ship curves in the direction *opposite* the direction of spin of the gyro, the sign of $\dot{\psi}$ must be changed. Then (80·3) becomes

$$AIr^4+(AK+Ik)r^3+[C^2\omega^2-C\omega\dot{\psi}(A+I)+AWh+Iwl+Kk]r^2$$
$$+[-C\omega\dot{\psi}(K+k)+Kwl+kWh]r$$
$$+[C^2\omega^2\dot{\psi}^2-C\omega\dot{\psi}(Wh+wl)+Whwl]=0. \quad (80\cdot 6)$$

In this case the first condition for stability requires that

$$C\omega\dot{\psi}(A+I)<(C^2\omega^2+AWh+Iwl+Kk),$$
$$C\omega\dot{\psi}(K+k)<(Kwl+kWh)$$

and
$$C\omega\dot{\psi}(Wh+wl)<(C^2\omega^2\dot{\psi}^2+Whwl).$$

Changing the sign of $\dot{\psi}$ also causes a change of sign in some of the terms in E, F, S and P, and therefore affects the angle of heel of the ship and the amplitude of the roll. The behavior of a ship is thus less certain when its path is curving in the direction opposite the direction of spin of the gyro.

81. Active-type stabilizer

To find the differential equations of motion under the action of the Sperry stabilizer, we put $w=0$ and $k=0$ in equations (80·1) and then make the assumptions of Art. 79. The equations are then

$$\left.\begin{aligned}
&(a)\quad I\ddot{\phi}+K\dot{\phi}-C\omega\dot{\theta}+(C\omega\dot{\psi}+Wh)\phi=\frac{-pWV\dot{\psi}}{g}+M\sin bt,\\
&(b)\quad A\ddot{\theta}+C\omega\dot{\phi}+C\omega\dot{\psi}\theta=R(t),\\
&(c)\quad \dot{\theta}=-m\dot{\phi}.
\end{aligned}\right\} \quad (81\cdot 1)$$

On substituting (c) into (a) we get

$$I\ddot{\phi}+(K+C\omega m)\dot{\phi}+(Wh+C\omega\dot{\psi})\phi=\frac{-pWV\dot{\psi}}{g}+M\sin bt. \quad (81\cdot 2)$$

Here the characteristic equation is

$$Ir^2+(K+C\omega m)r+Wh+C\omega\dot{\psi}=0,$$

whose roots are

$$r=\frac{-(K+C\omega m)}{2I}\pm\frac{\sqrt{\{(K+C\omega m)^2-4I(Wh+C\omega\dot{\psi})\}}}{2I}.$$

If $(K+C\omega m)^2 > 4I(Wh+C\omega\dot{\psi})$, both roots will be real and negative. The motion will be overdamped and there will be no natural oscillation.

If $(K+C\omega m)^2 < 4I(Wh+C\omega\dot{\psi})$, the values of r will be

$$r = \frac{-(K+C\omega m)}{2I} \pm i\frac{\sqrt{\{4I(Wh+C\omega\dot{\psi})-(K+C\omega m)^2\}}}{2I}.$$

In this case the free motion of the ship is a damped roll of period

$$T = \frac{4\pi I}{\sqrt{\{4I(Wh+C\omega\dot{\psi})-(K+C\omega m)^2\}}},$$

which could be very long.

If the ship is turning in the direction *opposite* the direction of spin of the gyro, $\dot{\psi}$ will be changed in sign. The characteristic equation in this case becomes

$$Ir^2 + (K+C\omega m)r + Wh - C\omega\dot{\psi} = 0.$$

If the constant term $(Wh - C\omega\dot{\psi})$ is negative, that is if $C\omega\dot{\psi} > Wh$, both values of r will be real and one of them will be positive. Hence the motion will be unstable in this case.

To find a particular solution of (81·2), we put

$$\phi = P_1 + Q_1 \sin bt + Q_2 \cos bt.$$

Then

$$\dot{\phi} = Q_1 b \cos bt - Q_2 b \sin bt,$$

$$\ddot{\phi} = -Q_1 b^2 \sin bt - Q_2 b^2 \cos bt.$$

Substituting these into (81·2) and then equating coefficients of like functions on the two sides of the equation, we get

$$P_1 = \frac{-pWV\dot{\psi}}{g(Wh+C\omega\dot{\psi})},$$

$$(Wh + C\omega\dot{\psi} - Ib^2)Q_1 - b(K+C\omega m)Q_2 = M,$$

$$b(K+C\omega m)Q_1 + (Wh + C\omega\dot{\psi} - Ib^2)Q_2 = 0.$$

Solving for Q_1 and Q_2, we find

$$Q_1 = \frac{M(Wh+C\omega\dot{\psi}-Ib^2)}{(Wh+C\omega\dot{\psi}-Ib^2)^2 + b^2(K+C\omega m)^2},$$

$$Q_2 = \frac{-Mb(K+C\omega m)}{(Wh+C\omega\dot{\psi}-Ib^2)^2 + b^2(K+C\omega m)^2}.$$

A particular solution of (81·2) is therefore

$$\phi=\frac{-pWV\dot{\psi}}{g(Wh+C\omega\dot{\psi})}+\frac{M}{(Wh+C\omega\dot{\psi}-Ib^2)^2+b^2(K+C\omega m)^2}$$
$$\times[(Wh+C\omega\dot{\psi}-Ib^2)\sin bt-b(K+C\omega m)\cos bt],$$

which can be written in the form

$$\phi=\frac{-pWV\dot{\psi}}{g(Wh+C\omega\dot{\psi})}+\frac{M}{\sqrt{\{(Wh+C\omega\dot{\psi}-Ib^2)^2+b^2(K+C\omega m)^2\}}}\sin(bt-\gamma), \tag{81·3}$$

where $$\gamma=\tan^{-1}\frac{b(K+C\omega m)}{Wh+C\omega\dot{\psi}-Ib^2}.$$

If the ship is turning in a direction *opposite* the direction of spin of the gyro, the sign of $\dot{\psi}$ must be changed. Equation (81·3) then becomes

$$\phi=\frac{pWV\dot{\psi}}{g(Wh-C\omega\dot{\psi})}+\frac{M}{\sqrt{\{(Wh-C\omega\dot{\psi}-Ib^2)^2+b^2(K+C\omega m)^2\}}}\sin(bt-\delta), \tag{81·4}$$

where $$\delta=\tan^{-1}\frac{b(K+C\omega m)}{Wh-C\omega\dot{\psi}-Ib^2}.$$

It will be seen that in this case the ship is heeled inward toward the center of curvature of the path and at a greater angle of inclination than when it is turning in the same direction as the spin of the gyro. The amplitude and phase angle of the roll have also changed.

If we put $\dot{\psi}=0$ in (81·3) and (81·4), we get (79·4). The amplitude and phase of roll on a curved path, aside from the heel, are thus not the same as on a straight course.

The unsymmetrical effect of stabilizers on curves could be corrected by using two identical gyros rigidly connected by toothed gears somewhat in the manner shown in Fig. 65. If both gyros rotated in opposite directions at the same speed and precessed in opposite directions at the same speed, the one-sided effect of either gyro when traversing a curve would be annulled by the other gyro.

CHAPTER IX

The Gyroscope as a Stabilizer (Continued) Monorail Cars

A. MONORAIL CARS ON A STRAIGHT TRACK

Railway cars designed to run on a single rail, with the center of gravity above the rail and held in dynamical equilibrium by gyroscopic action, were invented during the first decade of the present century. The first car of this kind was invented by Louis Brennan of England about 1905, and further developed and improved by him in subsequent years. On this car the gyro was mounted with its spin axis horizontal and at right angles to the track (rail).

In 1909 August Scherl of Germany brought out a monorail car on which the gyro was mounted with its spin axis vertical, as in the case of the ship stabilizer. The Scherl type of car was studied and possibly improved a little later by P. P. Schilowsky of Russia.* More recently (1952) a monorail car of improved design has been demonstrated in Germany.

82. Monorail car with gyro axis vertical (Scherl and Schilowsky type)

The type with vertical spin axis is represented schematically in Fig. 63. This type of car has a weight mounted above the trunnion axis so as to make the center of gravity of gyro and frame lie *above* the trunnion axis, thus making the combined gyro and frame gravitationally unstable about the trunnion axis. The car and mounted gyroscope thus constitute an inverted gyroscopic pendu-

* 'The present writer had a considerable advantage over both Brennan and Scherl, in that he entered the field after them and was therefore in a position to observe and avoid all the deficiencies of their systems.'—P. P. Schilowsky, *The Gyroscope: Its Practical Construction and Application* (1924), p. 137.

lum of the type treated in Art. 74. To find the differential equations of motion, let

W = weight of car and all equipment,

w = weight of gyro and frame,

h = height of c.g. of car and equipment above rail,

l = height of c.g. of gyro and frame above trunnion axis,

I = moment of inertia of car and equipment about rail,

A = moment of inertia of gyro and frame about trunnion axis.

Then

$Wh \sin\phi = Wh\phi$ = moment of W about rail,

$wl \sin\theta = wl\theta$ = moment of w about trunnion axis,

$K\dot{\phi}$ = moment of friction about rail,

$k\dot{\theta}$ = damping moment about trunnion axis.

Hence with axes as in Fig. 56 we have

$\omega_{x'} = -\dot{\phi}\cos\theta = -\dot{\phi}$ approximately, $\omega_{y'} = \dot{\theta}$, $\omega_{z'} = -\dot{\phi}\sin\theta$;

$H_{x'} = -I\dot{\phi}\cos\theta = -I\dot{\phi}$, practically;

$H_{y'} = A\dot{\theta}$, $H_{z'} = C(\omega - \dot{\phi}\sin\theta) = C\omega$, practically.

Then $$\frac{dH_{x'}}{dt} = -I\ddot{\phi}, \quad \frac{dH_{y'}}{dt} = A\ddot{\theta}, \quad \frac{dH_{z'}}{dt} = 0.$$

Substituting the above quantities into (16·4) and keeping in mind the directions of the moments, we have

$$-I\ddot{\phi} + C\omega\dot{\theta} - A\dot{\theta}(\dot{\phi}\sin\theta) = -Wh\phi + K\dot{\phi},$$

$$A\ddot{\theta} + (-I\dot{\phi})(-\dot{\phi}\sin\theta) - C\omega(-\dot{\phi}\cos\theta) = wl\theta - k\dot{\theta},$$

$$A\dot{\theta}(-\dot{\phi}\cos\theta) - (-I\dot{\phi}\cos\theta)\,\dot{\theta} = 0.$$

Since $\dot{\phi}$, $\dot{\theta}$ and $\sin\theta$ are all small, we neglect the products containing $\dot{\theta}\dot{\phi}$ and put $\cos\theta = 1$. Then we have

$$\left.\begin{aligned} &(a) \quad I\ddot{\phi} - C\omega\dot{\theta} + K\dot{\phi} = Wh\phi, \\ &(b) \quad A\ddot{\theta} + C\omega\dot{\phi} + k\dot{\theta} = wl\theta. \end{aligned}\right\} \tag{82·1}$$

These equations can also be derived by the G.R.M. method as follows:*

A forced precession of angular velocity $\dot{\phi}$ about the rail induces a G.R.M. of magnitude $C\omega\dot{\phi}$ about the trunnion axis, and this G.R.M. is negative (its vector points in the negative direction of the y-axis).

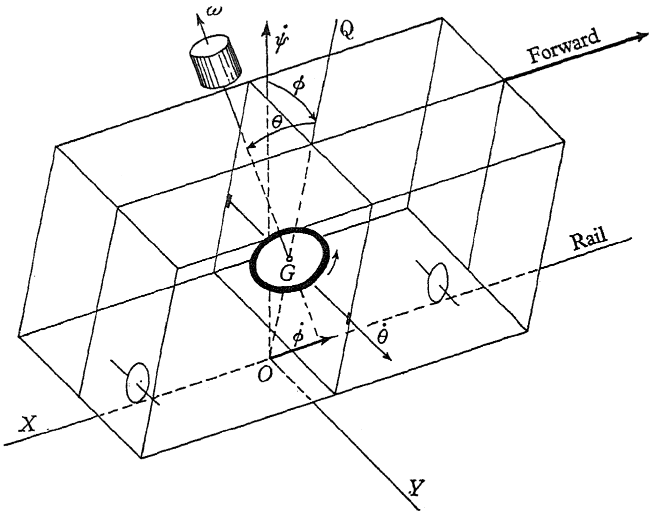


Fig. 63

Then the forced precession about the transverse axis in the negative direction induces a G.R.M. of $C\omega(-\dot{\theta})$ about the rail axis in the backward (negative in this case) direction. That is, the G.R.M. in the direction of the rail is $-C\omega(-\dot{\theta})=C\omega\dot{\theta}$. Then by the fundamental equation of rotation about an axis (Art. 20) we have:

$$I\ddot{\phi}=Wh\phi-K\dot{\phi}+C\omega\dot{\theta},$$

$$A\ddot{\theta}=wl\theta-k\dot{\theta}-C\omega\dot{\phi},$$

or

$$I\ddot{\phi}+K\dot{\phi}-C\omega\dot{\theta}=Wh\phi,$$

$$A\ddot{\theta}+k\dot{\theta}+C\omega\dot{\phi}=wl\theta,$$

which are the same as (82·1).

* Note that the forward direction of motion is in the negative direction of the x-axis and that, as ϕ increases, the velocity vector $\dot{\phi}$ points in the negative x-direction.

To find the characteristic (auxiliary) equation for the system (82·1), we assume

$$\phi = ae^{rt}, \qquad \theta = be^{rt}.$$

Then

$$\dot{\phi} = are^{rt}, \qquad \dot{\theta} = bre^{rt},$$

$$\ddot{\phi} = ar^2e^{rt}, \qquad \ddot{\theta} = br^2e^{rt}.$$

Substituting these into (82·1) and then dividing the results throughout by e^{rt}, we get

$$(Ir^2 + Kr - Wh)\,a - C\omega rb = 0,$$

$$C\omega ra + (Ar^2 + kr - wl)\,b = 0.$$

These homogeneous equations in a and b will have a common solution if, and only if, the determinant of their coefficients is equal to zero. Hence we have

$$\begin{vmatrix} Ir^2 + Kr - Wh & -C\omega r \\ C\omega r & Ar^2 + kr - wl \end{vmatrix} = 0.$$

Expanding and arranging in descending powers of r, we get

$$AIr^4 + (KA + kI)\,r^3 + (C^2\omega^2 - AWh - Iwl + Kk)\,r^2 + (-Whk - wlK)\,r + Whwl = 0. \quad (82\cdot2)$$

A first condition for the stability of the monorail car is that all the coefficients and the constant term in (82·2) shall be positive. This means, first of all, that l and h must have like signs, which means that the center of gravity of the gyro and frame must be *above* the trunnion axis.

Since the moment of rail friction, $K\dot{\phi}$, is necessarily a retarding moment, the coefficient K is necessarily positive and the sign of the product wlK cannot be changed. In order that the coefficient of r be positive, the term Whk must be changed in sign and must be numerically greater than wlK. This means that the moment $k\dot{\theta}$, which we assumed to be a retarding moment, must actually be an *accelerating* moment. Hence if the monorail car of the Scherl type is to be stable, there must be some provision for accelerating the precession about the trunnion axis. Scherl surmounted this difficulty by installing a servomotor to apply automatically an accelerating moment about that axis, as did Sperry in the case of his ship stabilizer. We therefore change the differential equations (82·1) so as to provide for the accelerating moment.

Let us assume that the servomotor applies an accelerating moment of magnitude $R\phi$ about the trunnion axis, where R denotes a positive constant. Then since there is no braking moment $k\dot{\theta}$ (we neglect bearing friction), equations (82·1) become

$$\left.\begin{aligned} I\ddot{\phi}+K\dot{\phi}-C\omega\dot{\theta}-Wh\phi=0, \\ A\ddot{\theta}+C\omega\dot{\phi}-wl\theta=-R\phi. \end{aligned}\right\} \qquad (82\cdot3)$$

The characteristic equation for (82·3) is found as above by putting

$$\phi=ae^{rt}, \quad \theta=be^{rt}.$$

The result is

$$AIr^4+AKr^3+(C^2\omega^2-AWh-Iwl)\,r^2+(C\omega R-Kwl)\,r+Whwl=0. \qquad (82\cdot4)$$

These coefficients will all be positive if

$$C^2\omega^2>(AWh+Iwl),$$

and

$$C\omega R>Kwl.$$

These conditions can evidently be satisfied by increasing $C\omega$ sufficiently.

Because of the fact that the gyro in a monorail car must spin at very high speed (higher than in the case of the ship stabilizer), we may treat the quantity $AWh+Iwl$ as negligible in comparison with $C^2\omega^2$. On neglecting this quantity, equation (82·4) becomes

$$AIr^4+AKr^3+C^2\omega^2r^2+(C\omega R-Kwl)\,r+Whwl=0. \qquad (82\cdot5)$$

We know from our previous studies of the gyroscope that two roots of (82·5) are small and that the other two roots are considerably larger. We get approximate values of the larger roots by neglecting the last two terms of (82·5) and solving the quadratic equation

$$AIr^2+AKr+C^2\omega^2=0,$$

from which we get

$$r=-\frac{K}{2I}\pm i\frac{\sqrt{(4AIC^2\omega^2-A^2K^2)}}{2AI},$$

if $A^2K^2<4AIC^2\omega^2$. We may further neglect the second term under the radical in comparison with the first term. Then we have

$$r=-\frac{K}{2I}\pm i\frac{C\omega}{\sqrt{(AI)}}.$$

Here the period of oscillation is

$$T_1 = \frac{2\pi\sqrt{(AI)}}{C\omega},$$

which may be very short.

To get the two smaller roots of (82·5), we neglect the first two terms and solve the quadratic

$$C^2\omega^2 r^2 + (C\omega R - Kwl)\,r + Whwl = 0,$$

which gives

$$r = \frac{-(C\omega R - Kwl)}{2C^2\omega^2} \pm i\frac{\sqrt{\{4C^2\omega^2 Whwl - (C\omega R - Kwl)^2\}}}{2C^2\omega^2},$$

if $(C\omega R - Kwl)^2 < 4C^2\omega^2 Whwl$.

If we neglect the quantity $(C\omega R - Kwl)^2$ in comparison with $4C^2\omega^2 Whwl$, the value of r becomes

$$r = \frac{-(C\omega R - Kwl)}{2C^2\omega^2} \pm i\frac{\sqrt{(Whwl)}}{C\omega}.$$

In this solution the damping coefficient is small and the period

$$T_2 = \frac{2\pi C\omega}{\sqrt{(Whwl)}}$$

is long.

It is to be noted that if $C\omega R > Kwl$, the real parts of all the roots of (82·5) are negative and the motion of the monorail car is therefore stable.

83. Monorail car with gyro axis horizontal (Brennan type)

The Brennan monorail car is represented schematically in Fig. 64. The gyro is mounted with its spin axis horizontal and perpendicular to the rail, as previously stated. The frame is made unstable about the precession axis by application of an unstabilizing moment proportional to θ. This unstabilizing moment is applied automatically by a mechanical device and corresponds to $wl\theta$ in Art. 82, the coefficient of θ having the same sign as Wh. In the following treatment of the Brennan car we shall use the same notation as in Art. 82, even to the point of using wl for the coefficient of the unstabilizing moment acting on the frame. We shall derive the differential equations of motion by the G.R.M. method (see footnote on p. 217).

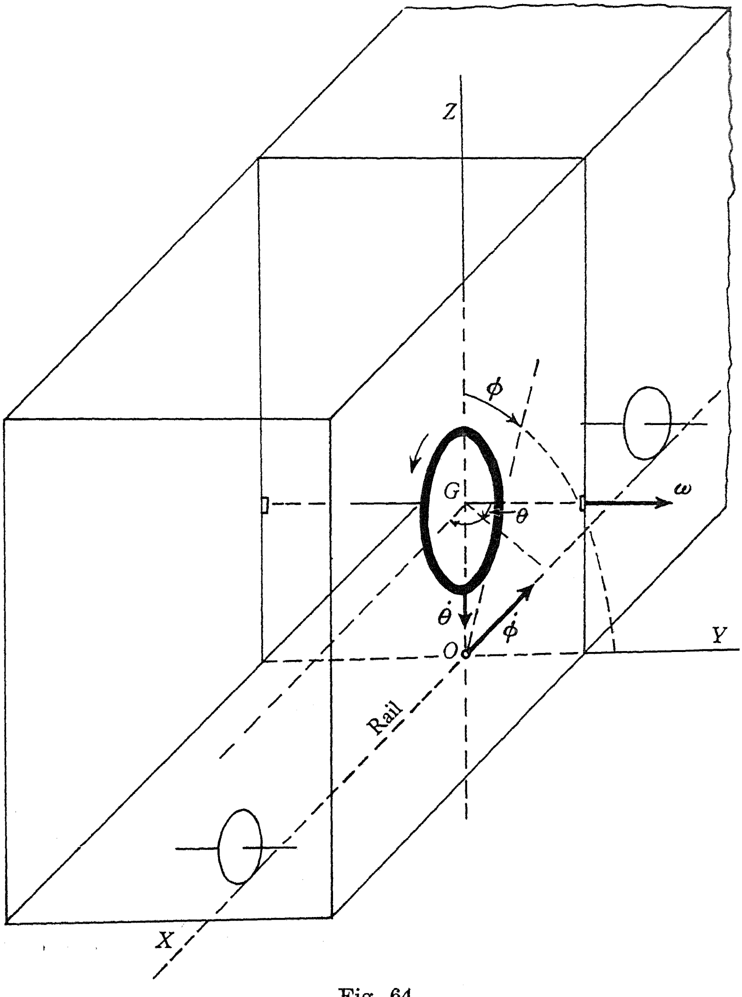


Fig. 64

Referring to Fig. 64, we see that if the car tilts to the right with angular velocity $\dot{\phi}$, this is a forced precession about the rail axis. Hence this forced precession induces a G.R.M. of $C\omega\dot{\phi}$ about the vertical axis OZ and in a counterclockwise (negative) direction as viewed from above, or $-C\omega\dot{\phi}$.

The forced precession about OZ in the negative direction induces a G.R.M. of $C\omega(-\dot{\theta})$ about a line parallel to the rail and directed

backward, that is, in the negative direction. Hence the G.R.M. in this case is $-C\omega(-\dot{\theta})$, or $C\omega\dot{\theta}$. Then from the fundamental equation of rotation about an axis (Art. 20) we have

$$I\ddot{\phi} = Wh\phi - K\dot{\phi} + C\omega\dot{\theta},$$

and

$$A\ddot{\theta} = wl\theta + k\dot{\theta} - C\omega\dot{\phi} \quad \text{(here we take } k\dot{\theta} \text{ to be an } \textit{accelerating} \text{ moment)},$$

or

$$\left.\begin{aligned} &(a) \quad I\ddot{\phi} + K\dot{\phi} - C\omega\dot{\theta} = Wh\phi, \\ &(b) \quad A\ddot{\theta} - k\dot{\theta} + C\omega\dot{\phi} = wl\theta. \end{aligned}\right\} \qquad (83\cdot1)$$

On putting $\phi = ae^{rt}$, $\theta = be^{rt}$ and proceeding as in Art. 82, we find the characteristic equation for the system (83·1) to be

$$AIr^4 + (AK - Ik)r^3 + (C^2\omega^2 - AWh - Iwl - Kk)r^2 + (Whk - wlK)r + Whwl = 0. \qquad (83\cdot2)$$

All coefficients in (83·2) will be positive if

$$AK > Ik, \quad Whk > wlK,$$

and

$$C^2\omega^2 > Kk + AWh + Iwl.$$

Since $I > A$, the above conditions lead to the further conditions

$$k < K \quad \text{and} \quad \frac{wl}{Wh} < \frac{A}{I}.$$

Because of the fact that the gyro in a monorail car must spin with very high speed (higher than in the case of the ship stabilizer), we may assume that the sum of terms $Kk + AWh + Iwl$ is negligible in comparison with $C^2\omega^2$. When those terms are neglected, equation (83·2) becomes

$$AIr^4 + (AK - Ik)r^3 + C^2\omega^2r^2 + (Whk - wlK)r + Whwl = 0. \qquad (83\cdot3)$$

We know from our previous studies of the gyroscope that two roots of (83·3) are small and that the other two are considerably larger. We get approximate values of the larger roots by neglecting the last two terms of (83·3) and solving the quadratic equation

$$AIr^2 + (AK - Ik)r + C^2\omega^2 = 0,$$

from which we get

$$r = \frac{-(AK - Ik) \pm \sqrt{\{(AK - Ik)^2 - 4AIC^2\omega^2\}}}{2AI}.$$

Now treating the quantity $(AK - Ik)^2$ as negligible in comparison with $4AIC^2\omega^2$, we get

$$r = -\frac{(AK - Ik)}{2AI} \pm i \frac{C\omega}{\sqrt{(AI)}}. \tag{83·4}$$

To get approximate values of the smaller roots of (83·3) we neglect the first two terms and solve the quadratic

$$C^2\omega^2 r^2 + (Whk - wlK) r + Whwl = 0.$$

From this we get

$$r = \frac{-(Whk - wlK) \pm \sqrt{\{(Whk - wlK)^2 - 4C^2\omega^2 Whwl\}}}{2C^2\omega^2}.$$

We now neglect the quantity $(Whk - wlK)^2$ in comparison with $4C^2\omega^2 Whwl$ and thus obtain

$$r = \frac{-(Whk - wlK)}{2C^2\omega^2} \pm i \frac{\sqrt{(Whwl)}}{C\omega}. \tag{83·5}$$

The values of r given by (83·4) and (83·5) show that if the real parts of the complex roots are to be negative, we must have

$$AK > Ik \quad \text{from (83·4)},$$

and

$$Whk > wlK \quad \text{from (83·5)},$$

as previously stated on the basis of Routh's conditions.

The periods of the two types of oscillations are

$$T_1 = \frac{2\pi C\omega}{\sqrt{(Whwl)}} \quad \text{and} \quad T_2 = \frac{2\pi\sqrt{(AI)}}{C\omega}.$$

Hence the long period (T_1) can be lengthened by increasing ω or by decreasing l; and the short period can be shortened by increasing ω or by decreasing A.

B. MONORAIL CARS ON CURVES

84. Vertical axis type

Since the gyro on a monorail car of vertical axis (of spin) type is mounted in the same manner as the gyro of a ship stabilizer, the differential equations of motion of such a car on a curve will be very similar to those for a stabilized ship on a curve. We find them by substituting the appropriate quantities into equations (16·4). The left-hand members will be the same as for the ship stabilizer in Art. 80. To find the moments M_x and M_y, let $R\phi$ denote the accelerating moment about the trunnion axis, where R denotes a positive constant. Then the moment about the precession axis will be

$$M_y = wl\theta - R\phi.$$

Since the centrifugal force tends to tilt the car outward away from the center of curvature, the moment about the x-axis will be

$$M_x = -Wh\phi - \frac{pWV\dot{\psi}}{g} + K\dot{\phi}.$$

The differential equations of motion for the Scherl monorail car running on a curve are therefore

$$-I\ddot{\phi} + C\omega\dot{\theta} - C\omega\dot{\psi}\phi = -Wh\phi - \frac{pWV\dot{\psi}}{g} + K\dot{\phi},$$
$$A\ddot{\theta} + C\omega\dot{\phi} + C\omega\dot{\psi}\theta = wl\theta - R\phi,$$

or

$$\left.\begin{aligned} &(a)\quad I\ddot{\phi} + K\dot{\phi} - C\omega\dot{\theta} + (C\omega\dot{\psi} - Wh)\phi = \frac{pWV\dot{\psi}}{g}, \\ &(b)\quad A\ddot{\theta} + C\omega\dot{\phi} + (C\omega\dot{\psi} - wl)\theta + R\phi = 0. \end{aligned}\right\} \qquad (84{\cdot}1)$$

To solve these equations for ϕ, we first differentiate (b) and get

$$(c)\quad A\dddot{\theta} + C\omega\ddot{\phi} + (C\omega\dot{\psi} - wl)\dot{\theta} + R\dot{\phi} = 0.$$

Solving (a) for $\dot{\theta}$ and then differentiating the result, we have

$$\dot{\theta} = \frac{1}{C\omega}\left[I\ddot{\phi} + K\dot{\phi} + (C\omega\dot{\psi} - Wh)\phi - \frac{pWV\dot{\psi}}{g}\right],$$

$$\ddot{\theta} = \frac{1}{C\omega}[I\dddot{\phi} + K\ddot{\phi} + (C\omega\dot{\psi} - Wh)\dot{\phi}],$$

$$\dddot{\theta} = \frac{1}{C\omega}[I\phi^{\text{iv}} + K\dddot{\phi} + (C\omega\dot{\psi} - Wh)\ddot{\phi}].$$

Substituting these into (c), we get

$$AI\phi^{\text{iv}}+AK\dddot{\phi}+[C^2\omega^2+C\omega\dot{\psi}(A+I)-(AWh+Iwl)]\ddot{\phi}$$
$$+(C\omega K\dot{\psi}+C\omega R-Kwl)\dot{\phi}+[C^2\omega^2\dot{\psi}^2-C\omega\dot{\psi}(Wh+wl)$$
$$+Whwl]\phi=\frac{pWV\dot{\psi}}{g}(C\omega\dot{\psi}-wl). \quad (84\cdot2)$$

Here the characteristic equation is

$$AIr^4+AKr^3+[C^2\omega^2+C\omega\dot{\psi}(A+I)-(AWh+Iwl)]r^2$$
$$+(C\omega K\dot{\psi}+C\omega R-Kwl)r$$
$$+[C^2\omega^2\dot{\psi}^2-C\omega\dot{\psi}(Wh+wl)+Whwl]=0. \quad (84\cdot3)$$

For stability, all coefficients must be positive and we must therefore have

$$C^2\omega^2+C\omega\dot{\psi}(A+I)>(AWh+Iwl),$$
$$C\omega K\dot{\psi}+C\omega R>Kwl,$$
$$C^2\omega^2\dot{\psi}^2+Whwl>C\omega\dot{\psi}(Wh+wl).$$

These conditions can be met by making $C\omega$ sufficiently large.

To find a particular solution of (84·2) and thus the amplitude of oscillation, we put

$$\phi=Q, \quad \text{a constant.}$$

Then

$$\dot{\phi}=\ddot{\phi}=\dddot{\phi}=\phi^{\text{iv}}=0.$$

Substituting these into (84·2), we get

$$Q=\phi=\frac{pWV(C\omega\dot{\psi}^2-wl\dot{\psi})}{g[C^2\omega^2\dot{\psi}^2-C\omega\dot{\psi}(Wh+wl)+Whwl]}.$$

We have already found that the denominator must be positive if the car is to be stable. Hence ϕ will be negative only when

$$C\omega\dot{\psi}<wl.$$

This means that the car will lean inward toward the center of curvature only when $C\omega\dot{\psi}<wl$.

A necessary condition for the dynamical equilibrium of the car is that the resultant of the centrifugal force and the weight of the car must pass through the rail. The requirement that $C\omega\dot{\psi}<wl$ puts a severe limitation on the velocity of spin of the gyro and may be sufficient to explain the failure of the Scherl car to run satisfactorily

on curves. The defect was corrected by using two gyros of the same size, spinning and precessing at the same speeds but in opposite directions, and having their frames rigidly connected by toothed gears (see Fig. 65).

If the car is running on a track which curves in the direction *opposite* the direction of spin of the gyro, the sign of $\dot{\psi}$ must be changed in the preceding equations. When the sign of $\dot{\psi}$ is changed, the characteristic equation (84·3) becomes

$$AIr^4 + AKr^3 + \{C^2\omega^2 - [C\omega\dot{\psi}(A+I) + AWh + Iwl]\}r^2 + [C\omega R - (C\omega K\dot{\psi} + Kwl)]r + [C^2\omega^2\dot{\psi}^2 + C\omega\dot{\psi}(Wh+wl) + Whwl] = 0.$$

Here the conditions for stability are

$$C^2\omega^2 > [C\omega\dot{\psi}(A+I) + AWh + Iwl],$$
$$C\omega R > (C\omega K\dot{\psi} + Kwl),$$

which can be met by making $C\omega$ and R large enough.

The angle of inclination in this case becomes

$$\phi = \frac{pWV(C\omega\dot{\psi}^2 + wl\dot{\psi})}{g[C^2\omega^2\dot{\psi}^2 + C\omega\dot{\psi}(Wh+wl) + Whwl]}$$

and is definitely positive. The car therefore always leans toward the centre of curvature of the track, making it always possible for the resultant of the centrifugal force and the weight of the car to pass through the rail and thus fulfill the fundamental condition for dynamical equilibrium.

The Schilowsky car was forced to precess by means of a pendulum.

85. Horizontal axis type

When the spin axis of a gyro is horizontal, a rotation in azimuth augments or diminishes its precession about a vertical axis, depending on the direction of the rotation. Hence when a monorail car of the Brennan type runs on a curved track, the rate of precession $\dot{\theta}$ is increased or decreased by the angular velocity with which the car changes its direction. Furthermore, the precession angle θ is also increased or decreased by the angle $\dot{\psi}t$. If the track is curving to the right (see Fig. 64), the angular velocity of precession will therefore be $\dot{\theta} + \dot{\psi}$ and the precession angle after time t will be

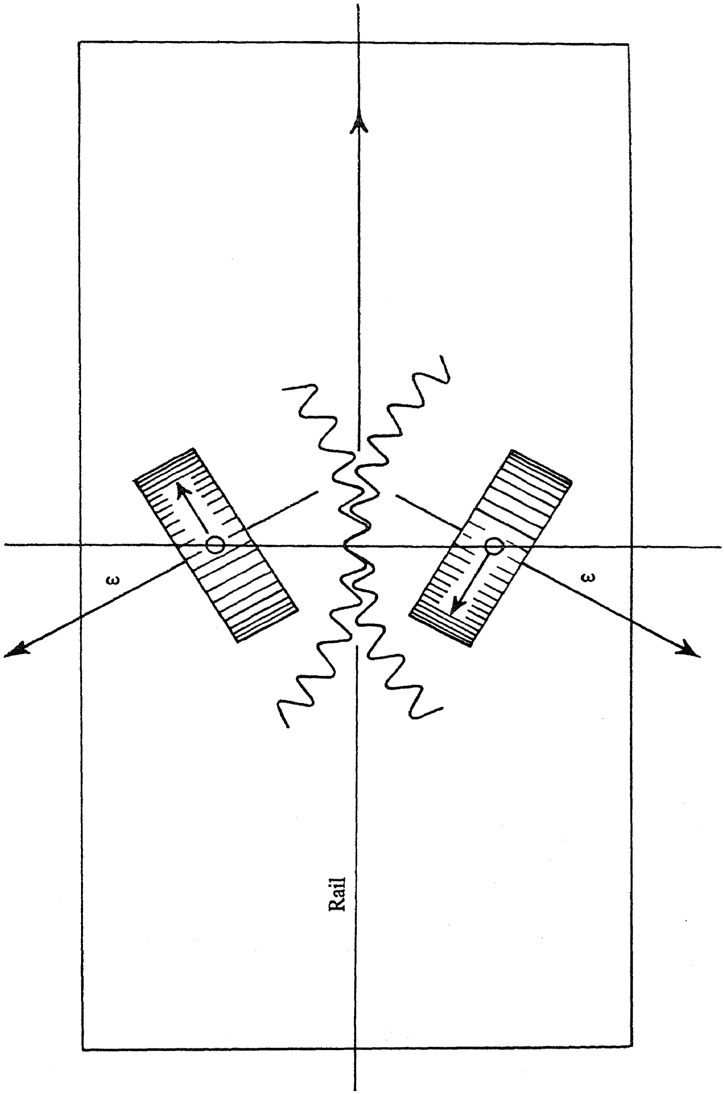


Fig. 65

$\theta+\dot{\psi}t$. We can therefore write the differential equations of motion for the Brennan car on a curve by replacing $\dot{\theta}$ by $\dot{\theta}+\dot{\psi}$ and replacing θ by $\theta+\dot{\psi}t$ in equations (83·1), and then adding to the right-hand side of (83·1 (a)) the moment of the centrifugal force. The differential equations of motion are therefore

$$\left.\begin{aligned} &(a)\quad I\ddot{\phi}+K\dot{\phi}-C\omega(\dot{\theta}+\dot{\psi})-Wh\phi=-\frac{WhV\dot{\psi}}{g},\\ &(b)\quad A\ddot{\theta}+k(\dot{\theta}+\dot{\psi})+C\omega\dot{\phi}-wl(\theta+\dot{\psi}t)=0. \end{aligned}\right\} \tag{85·1}$$

The term on the right in (85·1 (a)) is negative because the moment of the centrifugal force tends to overturn the car outward (to the left) and thus to decrease ϕ.

To solve equations (85·1) we first differentiate (b) with respect to time, thereby obtaining

$$(c)\quad A\dddot{\theta}+k\ddot{\theta}+C\omega\ddot{\phi}-wl(\dot{\theta}+\dot{\psi})=0.$$

Then from (a) we have

$$\dot{\theta}+\dot{\psi}=\frac{1}{C\omega}\left(I\ddot{\phi}+K\dot{\phi}-Wh\phi+\frac{WhV\dot{\psi}}{g}\right),$$

and

$$\ddot{\theta}=\frac{1}{C\omega}(I\dddot{\phi}+K\ddot{\phi}-Wh\dot{\phi}),$$

$$\dddot{\theta}=\frac{1}{C\omega}(I\phi^{\text{iv}}+K\dddot{\phi}-Wh\ddot{\phi}).$$

On substituting these into (c), we get

$$\begin{aligned} AI\phi^{\text{iv}}+(AK+Ik)\,\dddot{\phi}+(C^2\omega^2+Kk-AWh-Iwl)\,\ddot{\phi}\\ -(Whk+wlK)\,\dot{\phi}+Whwl\phi=\frac{WhwlV\dot{\psi}}{g}. \end{aligned}\tag{85·2}$$

To find a particular solution of this equation, we put

$$\phi=Q.$$

Then

$$\dot{\phi}=\ddot{\phi}=\dddot{\phi}=\phi^{\text{iv}}=0.$$

On substituting these into (85·2), we get

$$Q=\phi=\frac{V\dot{\psi}}{g}. \tag{85·3}$$

The car thus inclines inward toward the center of curvature.

To find the corresponding particular solution for θ, we substitute the above particular value of ϕ into (85·1 (a)), giving

$$\dot{\theta}+\dot{\psi}=0,$$

integration of which gives

$$\theta \doteq -\dot{\psi}t.$$

The precession of the frame about the vertical axis is therefore not oscillatory but increases with time. Hence the car would soon overturn if left to itself.

If the car is running on a curve that turns to the left, we find its behavior by changing the sign of $\dot{\psi}$. Then (85·3) becomes

$$\phi=-\frac{V\dot{\psi}}{g},$$

which shows that the car is tilted toward the center of curvature in this case also.

In order to make his monorail car traverse curves either to the right or to the left with equal facility and remain dynamically stable while doing so, Brennan equipped his car with two exactly similar gyros which rotated at the same speed and in opposite directions. The gyros were mounted side by side with their spin axes in the same straight line when the car was in equilibrium, as indicated in Fig. 65. The frames were coupled by means of spur-toothed gears so that they precessed at the same rate but in opposite directions. Whatever effect the azimuthal motion produced on one gyro was therefore annulled by the other. The car always leaned toward the center of curvature, however, because the centrifugal force acted on the whole car and equipment.

CHAPTER X

Astronomical Applications

86. Precession and nutation of the earth's axis

The earth may be regarded as an enormous gyroscope whose center of mass is constrained to move in an elliptic orbit around the sun. The form of the earth is an oblate ~~spheroid~~ whose equatorial diameter is about 27 miles greater than its polar diameter. The polar axis (axis of spin) is not perpendicular to the plane of the earth's orbit (plane of the ecliptic) but is inclined from the perpendicular by about $23\frac{1}{2}°$. The direction of the polar axis in space changes but very little (about $50''.4$ a year) as the earth proceeds in its orbit around the sun.

Because of the equatorial bulge and the inclination of the polar axis just mentioned, the resultant attraction of the sun or moon does not pass through the center of mass of the earth. The resultant attraction is therefore equivalent to a force acting through the center of mass and a couple which tends to straighten up the polar axis and make it perpendicular to the plane of the ecliptic (see Fig. 66). Since, however, the spinning earth is free to turn in any manner about its center of mass, this couple can do nothing but cause the spin axis to precess (with a slight nutation) around the perpendicular to the plane of the ecliptic, the type of motion studied in Chapter IV.

By reason of the fact that the moon is comparatively near to the earth (238,800 miles away) and the plane of its orbit is inclined only about 5° from the plane of the ecliptic, the precession due to the moon's attraction on the equatorial bulge is more than twice as great as that due to the sun.

To find a mathematical expression for the couple causing the precession and thence the differential equations for the precession and nutation of the polar axis, let us assume that the sun is the attracting body. Assume that the earth is at the winter solstice, so that the precessional couple has its maximum value. Take a set of

rectangular coordinate axes with the origin at the center of mass of the earth and with the x-axis along the line of intersection of the plane of the ecliptic by the plane of the earth's equator. Then a plane through the polar axis of the earth and the center of the sun will be perpendicular to the x-axis, and the line in which this plane intersects the plane of the equator may therefore be taken as the y-axis (see Fig. 67).

Let (x, y, z) be the position of any particle dm in the earth, and let $\boldsymbol{\rho}$ denote the position vector of the particle. Let $\mathbf{R}$ be the position vector of the sun and let $\mathbf{r}$ be the vector drawn from (x, y, z) to the sun, so that

$$\mathbf{R} = \boldsymbol{\rho} + \mathbf{r}. \tag{86·1}$$

Let M denote the mass of the sun and γ the constant of gravitation. Then by the law of universal gravitation the particle dm is attracted along the vector $\mathbf{r}$ with the force

$$dF = \frac{\gamma M dm}{r^2},$$

or, as a vector,

$$d\mathbf{F} = \frac{\gamma M dm}{r^2}\boldsymbol{\epsilon} = \frac{\gamma M dm}{r^3} r\boldsymbol{\epsilon} = \frac{\gamma M dm}{r^3}\mathbf{r},$$

where $\boldsymbol{\epsilon}$ denotes a unit vector along $\mathbf{r}$.

Now from (86·1)

$$\mathbf{r} = \mathbf{R} - \boldsymbol{\rho}.$$

And from Fig. 67

$$\boldsymbol{\rho} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

$$\mathbf{R} = OQ + QS$$

$$= \mathbf{j}R\cos\theta - \mathbf{k}R\sin\theta.$$

Hence

$$\mathbf{r} = \mathbf{j}R\cos\theta - \mathbf{k}R\sin\theta - \mathbf{i}x - \mathbf{j}y - \mathbf{k}z$$

$$= -\mathbf{i}x + \mathbf{j}(R\cos\theta - y) - \mathbf{k}(R\sin\theta + z).$$

Then

$$d\mathbf{F} = \frac{\gamma M dm}{r^3}\{-\mathbf{i}x + \mathbf{j}(R\cos\theta - y) - \mathbf{k}(R\sin\theta + z)\}.$$

The moment of this force about the origin is, by (12·2),

$$d\mathbf{M} = \boldsymbol{\rho} \times d\mathbf{F} = \frac{\gamma M dm}{r^3}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ -x & R\cos\theta - y & -(R\sin\theta + z) \end{vmatrix}$$

$$= -\mathbf{i}R(y\sin\theta + z\cos\theta) + \mathbf{j}Rx\sin\theta + \mathbf{k}Rx\cos\theta.$$

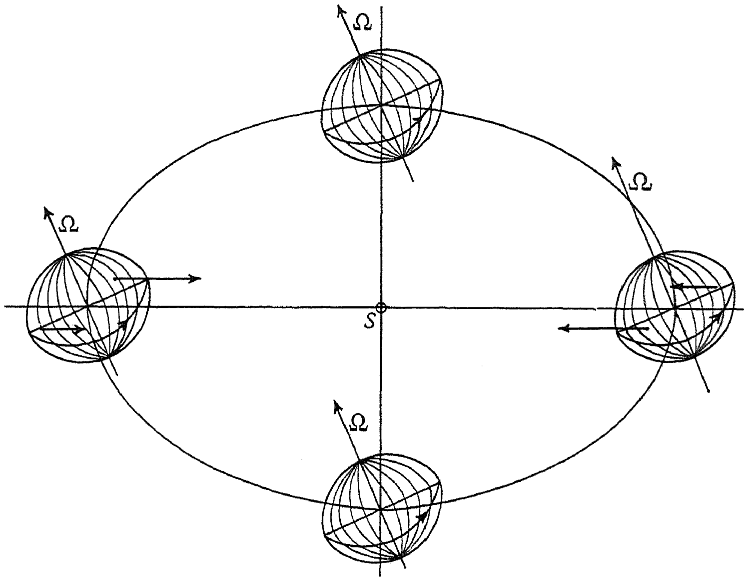


Fig. 66

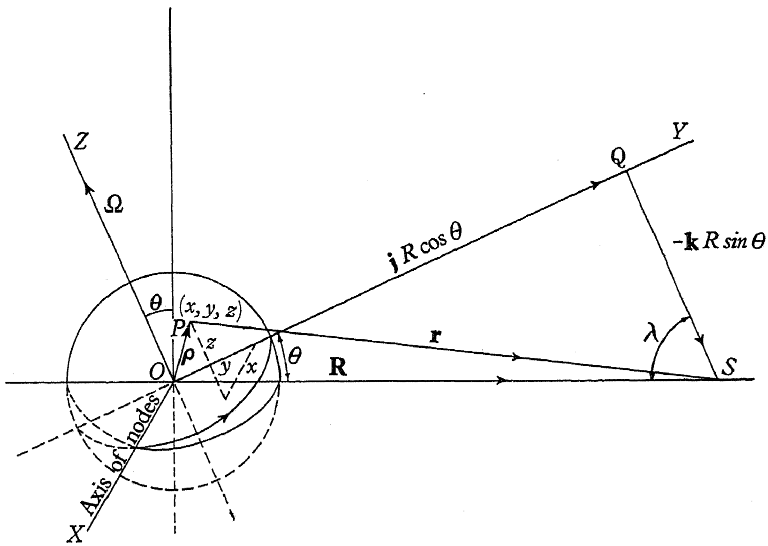


Fig. 67

Hence

$$\left.\begin{aligned} dM_x &= -\frac{\gamma RM\,dm}{r^3}(y\sin\theta + z\cos\theta), \\ dM_y &= \frac{\gamma RM\,dm}{r^3}\,x\sin\theta, \\ dM_z &= \frac{\gamma RM\,dm}{r^3}\,x\cos\theta. \end{aligned}\right\} \qquad (86{\cdot}2)$$

Since r is a function of x, y, z through (86·1), equations (86·2) cannot be integrated until r is expressed in terms of these variables. By the law of cosines we have

$$\begin{aligned} r^2 &= R^2 + \rho^2 - 2R\rho\cos(R,\rho) \\ &= R^2 + \rho^2 - 2\mathbf{R}\,.\,\boldsymbol{\rho} \\ &= R^2 + \rho^2 - 2(\mathbf{j}R\cos\theta - \mathbf{k}R\sin\theta)\cdot(\mathbf{i}x + \mathbf{j}y + \mathbf{k}z) \\ &= R^2 + \rho^2 - 2(Ry\cos\theta - Rz\sin\theta), \text{ by } (5{\cdot}6) \\ &= R^2\left[1 + \left(\frac{\rho}{R}\right)^2 - \frac{2}{R}(y\cos\theta - z\sin\theta)\right] \\ &= R^2\left[1 - \frac{2}{R}(y\cos\theta - z\sin\theta)\right], \end{aligned}$$

since $(\rho/R)^2$ is negligible in comparison with 1.* Hence

$$r = R\left[1 - \frac{2}{R}(y\cos\theta - z\sin\theta)\right]^{\frac{1}{2}},$$

and

$$\frac{1}{r^3} = \frac{1}{R^3}\left[1 - \frac{2}{R}(y\cos\theta - z\sin\theta)\right]^{-\frac{3}{2}}.$$

Now expanding the bracketed quantity into a binomial series and retaining only the first two terms of the series, we have

$$\frac{1}{r^3} = \frac{1}{R^3}\left[1 + \frac{3}{R}(y\cos\theta - z\sin\theta)\right].$$

* Even in the case of the moon the value of $(\rho/R)^2$ is $(\frac{3960}{238000})^2 = 0.000277$. In the case of the sun the quantity is far smaller than this.

On substituting this value of $1/r^3$ into equations (86·2) and integrating, we have

$$M_x = -\frac{\gamma M}{R^2}\left\{\int (y\sin\theta + z\cos\theta)\left[1 + \frac{3}{R}(y\cos\theta - z\sin\theta)\right]dm\right.$$

$$= -\frac{\gamma M}{R^2}\left\{\sin\theta\int y\,dm + \cos\theta\int z\,dm + \frac{3}{R}\left[\sin\theta\cos\theta\int (y^2 - z^2)\,dm\right.\right.$$

$$\left.\left. + \cos 2\theta\int yz\,dm\right]\right\}.$$

The first two integrals are zero because the origin is at the center of mass of the earth. The last integral, a product of inertia, is zero because the coordinate axes are principal axes of inertia. The value of M_x is thus

$$M_x = -\frac{3\gamma M}{R^3}\sin\theta\cos\theta\int (y^2 - z^2)\,dm$$

$$= -\frac{3\gamma M}{R^3}\sin\theta\cos\theta\int [x^2 + y^2 - (x^2 + z^2)]\,dm$$

$$= -\frac{3\gamma M}{R^3}\sin\theta\cos\theta\left[\int (x^2 + y^2)\,dm - \int (x^2 + z^2)\,dm\right].$$

The first integral on the right is the moment of inertia of the earth about the z-axis (polar axis), and the second integral is the moment of inertia about the y-axis (an equatorial axis). Denoting these moments of inertia by C and A, respectively, we get

$$M_x = -\frac{3\gamma M}{R^3}\sin\theta\cos\theta\,(C - A). \qquad (86\cdot3)$$

The values of M_y and M_z as found from (86·2) are zero, because they are given in terms of the integrals $\int x\,dm$, $\int xy\,dm$ and $\int xz\,dm$, each of which is zero for the reasons stated above. The moment of the attraction is thus entirely about the x-axis (axis of nodes) for the position of the earth here assumed—a result that was evident from the outset, because of the position of the coordinate axes. Equation (86·3) gives the maximum moment that can be exerted by the sun. A similar expression holds for the maximum moment that can be exerted by the moon.

Since M_x as given by (86·3) is a vector pointing in the negative direction along the x-axis, both because of its sign and because of

the right-handed-screw rule for the vector product $\boldsymbol{\rho}\times\mathbf{R}$, it may be written in the form

$$M_x = -\frac{3\mathbf{i}\gamma M}{R^3}(C-A)\sin\theta\cos\theta. \qquad (86{\cdot}4)$$

The presence of $\mathbf{i}$, $\sin\theta$ and $\cos\theta$ in (86·4) suggests that M_x may be expressed in a more general form as a vector dot product times a vector cross-product. Hence we multiply both numerator and denominator of (86·4) by R^2 and obtain

$$\begin{aligned}\mathbf{M} &= -\mathbf{i}\frac{3\gamma M}{R^5}(C-A)(R\sin\theta)(R\cos\theta)\\ &= \frac{3\gamma M}{R^5}(C-A)(R\cos\lambda)(-\mathbf{i}R\sin\lambda) \quad \text{(see Fig. 67)}\\ &= \frac{3\gamma M}{R^5}(C-A)(\mathbf{R}\cdot\mathbf{k})(\mathbf{R}\times\mathbf{k}). \qquad (86{\cdot}5)\end{aligned}$$

In order to find the moments about the axes of the moving trihedron and thence the differential equations of motion of the earth's axis, we must express the vector $\mathbf{M}$ in terms of unit vectors along the moving axes. This is accomplished by a process analogous to a transformation of coordinates. Referring to Fig. 68, let $O-X_1Y_1Z_1$ denote a set of fixed axes, with OX_1 and OY_1 lying in the plane of the ecliptic and OZ_1 being perpendicular to that plane. Let $O-XYZ$ denote the moving trihedron, OZ being the axis of spin (polar axis) of the earth, OX the line of intersection of the ecliptic plane by the plane of the earth's equator, and OY lying in the equatorial plane and at right angles to OX and OZ. If we assume that the moving axes originally coincided with the fixed set, they can be brought to the position shown in Fig. 68 by first rotating through the angle ψ about OZ_1, thus bringing OY_1 to the position OM; then rotating through the angle θ about OX, thereby bringing the y- and z-axes to the positions shown.

Let $\mathbf{i}_1$ and $\mathbf{j}_1$ denote unit vectors along OX_1 and OY_1, respectively, and let $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ denote unit vectors along the moving axes. Let $\mathbf{j}'$ denote a unit vector along OM. Then from the vector right triangles having unit vectors as hypotenuses, we have

$$\mathbf{i}_1 = OB = OA + AB = \mathbf{i}\cos\psi - \mathbf{j}'\sin\psi.$$

But $$\mathbf{j}' = OF = OE + EF = \mathbf{j}\cos\theta - \mathbf{k}\sin\theta.$$

Hence $$\begin{aligned}\mathbf{i}_1 &= \mathbf{i}\cos\psi - (\mathbf{j}\cos\theta - \mathbf{k}\sin\theta)\sin\psi\\ &= \mathbf{i}\cos\psi - \mathbf{j}\sin\psi\cos\theta + \mathbf{k}\sin\psi\sin\theta. \qquad (86{\cdot}6)\end{aligned}$$

And
$$\mathbf{j}_1 = OD = OC + CD = \mathbf{j}' \cos\psi + \mathbf{i}\sin\psi$$
$$= \mathbf{i}\sin\psi + (\mathbf{j}\cos\theta - \mathbf{k}\sin\theta)\cos\psi$$
$$= \mathbf{i}\sin\psi + \mathbf{j}\cos\psi\cos\theta - \mathbf{k}\cos\psi\sin\theta. \tag{86·7}$$

The position vector $\mathbf{R}$ changes in direction continuously as the earth travels around the sun. Let n denote the angular velocity of the earth in its orbit (assumed circular) around the sun. This is also

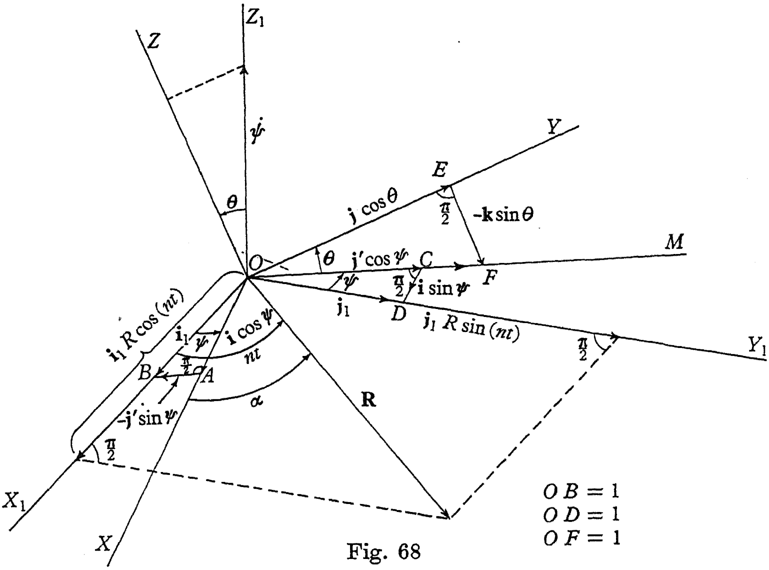


Fig. 68

the angular velocity of $\mathbf{R}$, which after a time t will have swept over an angle nt. Hence if $\mathbf{R}$ coincided with OX_1 at time $t = 0$, its position at time t will be given by the vector equation

$$\mathbf{R} = \mathbf{i}_1 R\cos nt + \mathbf{j}_1 R\sin nt.$$

Replacing $\mathbf{i}_1$ and $\mathbf{j}_1$ by their values as given in (86·6) and (86·7), we get

$$\begin{aligned}\mathbf{R} &= R\cos nt\,(\mathbf{i}\cos\psi - \mathbf{j}\sin\psi\cos\theta + \mathbf{k}\sin\psi\sin\theta)\\ &\qquad + R\sin nt\,(\mathbf{i}\sin\psi + \mathbf{j}\cos\psi\cos\theta - \mathbf{k}\cos\psi\sin\theta)\\ &= \mathbf{i}R\,(\cos nt\cos\psi + \sin nt\sin\psi)\\ &\qquad + \mathbf{j}R\cos\theta\,(\sin nt\cos\psi - \cos nt\sin\psi)\\ &\qquad - \mathbf{k}R\sin\theta\,(\sin nt\cos\psi - \cos nt\sin\psi)\\ &= \mathbf{i}R\cos(nt-\psi) + \mathbf{j}R\cos\theta\sin(nt-\psi) - \mathbf{k}R\sin\theta\sin(nt-\psi).\end{aligned}$$

Hence by (5·6)

$$\mathbf{R}\,.\,\mathbf{k} = -R\sin\theta\sin(nt-\psi);$$

and by (6·8)

$$\mathbf{R}\times\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ R\cos(nt-\psi) & R\cos\theta\sin(nt-\psi) & -R\sin\theta\sin(nt-\psi) \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \mathbf{i}R\cos\theta\sin(nt-\psi) - \mathbf{j}R\cos(nt-\psi).$$

Substituting into (86·5) these values of $\mathbf{R}\,.\,\mathbf{k}$ and $\mathbf{R}\times\mathbf{k}$, we have

$$\mathbf{M} = -\frac{3\gamma M}{R^5}(C-A)\,R\sin\theta\sin(nt-\psi)$$
$$\times[\mathbf{i}R\cos\theta\sin(nt-\psi) - \mathbf{j}R\cos(nt-\psi)].$$

Hence $$M_x = -\frac{3\gamma M}{R^3}(C-A)\sin\theta\cos\theta\sin^2(nt-\psi), \tag{86·8}$$

$$\left.\begin{aligned} M_y &= \frac{3\gamma M}{R^3}(C-A)\sin\theta\sin(nt-\psi)\cos(nt-\psi), \\ M_z &= 0. \end{aligned}\right\} \tag{86·9}$$

M_x and M_y can evidently be expressed in terms of double angles in the form

$$M_x = -\frac{3\gamma M}{2R^3}(C-A)\sin\theta\cos\theta\{1-\cos 2(nt-\psi)\}, \tag{86·8a}$$

$$M_y = \frac{3\gamma M}{2R^3}(C-A)\sin\theta\sin 2(nt-\psi). \tag{86·9a}$$

The angle $nt-\psi$ is the longitude of the sun. It is also the angular distance of the earth from the line of nodes. If we put $nt-\psi=\alpha$, equations (86·8) and (86·9) become

$$M_x = -\frac{3\gamma M}{R^3}(C-A)\sin\theta\cos\theta\sin^2\alpha, \tag{86·8b}$$

$$M_y = \frac{3\gamma M}{R^3}(C-A)\sin\theta\sin\alpha\cos\alpha. \tag{86·9b}$$

These equations show at a glance that the precession moment is zero for $\alpha=0$ and $\alpha=180°$; that is, when the earth is at either of the equinoctial points. M_x is always negative (always tending to make

the earth's axis perpendicular to the plane of the ecliptic) and has its greatest value when $\alpha=90°$ and $\alpha=270°$; that is, at the summer and winter solstices. M_y is zero at the equinoxes and solstices. It has its greatest values at $\alpha=45°$, $135°$, $225°$ and $315°$.

The mean value of M_x from $\alpha=0$ to $\alpha=2\pi$ is

$$\bar{M}_x = -\frac{3\gamma M}{2R^3}(C-A)\sin\theta\cos\theta \tag{86·10}$$

(since the mean value of $\sin^2\alpha$ for this interval is $\frac{1}{2}$), whereas the mean value of M_y over the same interval is zero.

To find the differential equations of motion of the polar axis we use equations (25·5), because $M_z=0$. From Fig. 68 we see at once that

$$\omega_x=\dot\theta \quad \text{and} \quad \omega_y=\dot\psi\sin\theta. \tag{86·11}$$

Then from (25·5), (86·8a) and (86·9a) we have

$$A\dot\omega_x - A\omega_y\omega_z + C\omega_y\Omega = -\frac{3\gamma M}{2R^3}(C-A)\sin\theta\cos\theta\{1-\cos 2(nt-\psi)\},$$

$$A\dot\omega_y + A\omega_x\omega_z - C\omega_x\Omega = \frac{3\gamma M}{2R^3}(C-A)\sin\theta\sin 2(nt-\psi),$$

where Ω denotes the angular velocity of the daily rotation of the earth. Since the angular velocity of the moving trihedron is so exceedingly slow (the axis of nodes skews around at only 50″.4 a year), the first two terms in the left-hand members of the above equations are negligible in comparison with the third terms. Hence on neglecting these terms and replacing ω_x and ω_y by the values given in (86·11), we have

$$C\Omega\dot\psi\sin\theta = -\frac{3\gamma M}{2R^3}(C-A)\sin\theta\cos\theta\{1-\cos 2(nt-\psi)\},$$

$$-C\Omega\dot\theta = \frac{3\gamma M}{2R^3}(C-A)\sin\theta\sin 2(nt-\psi),$$

or

$$\frac{d\psi}{dt} = -\frac{3\gamma M}{2\Omega R^3}\left(\frac{C-A}{C}\right)\cos\theta\{1-\cos 2(nt-\psi)\}, \tag{86·12}$$

$$\frac{d\theta}{dt} = -\frac{3\gamma M}{2\Omega R^3}\left(\frac{C-A}{C}\right)\sin\theta\sin 2(nt-\psi). \tag{86·13}$$

These are the differential equations for the precession and nutation of the earth's axis.

Equation (86·12) shows that $d\psi/dt$ is always negative and that ψ is therefore always decreasing. This means that the line of nodes is moving backward and thus the equinoxes have a retrograde motion. The rate of precession varies slightly and in a periodic manner owing to the term $\cos 2(nt-\psi)$. Its average value is found by omitting the periodic term $\cos 2(nt-\psi)$. Hence the average rate is

$$\left(\frac{d\psi}{dt}\right) = -\frac{3\gamma M}{2\Omega R^3}\left(\frac{C-A}{C}\right)\cos\theta. \tag{86·14}$$

The nutation is seen from (86·13) to be periodic, with a period of π radians per year. The spin axis of the earth thus traces on the celestial sphere an undulating curve whose mean arc distance is about $23\frac{1}{2}°$ from the pole of the ecliptic. The motion is a pseudo-regular precession and is very similar to that discussed in Arts. 37 and 38.

To find the numerical value of the annual precession due to the sun's attraction, we substitute the appropriate numerical data into (86·14). But since we do not know the numerical value of M, we must eliminate it.

In the revolution of the earth around the sun, the sun's gravitational attraction on the earth is balanced by the centrifugal force acting on the earth. Hence we have the equation

$$\frac{\gamma Mm}{R^2} = mRn^2, \tag{86·15}$$

where m denotes the mass of the earth and n denotes the angular velocity of revolution. From the familiar equation $\theta = nt$ for circular motion, we get

$$2\pi = nT \quad \text{or} \quad n = \frac{2\pi}{T},$$

where T denotes the period of revolution (time for describing the angle 2π). Substituting this value for n into (86·5) and dividing through by m, we get

$$\frac{\gamma M}{R^3} = \left(\frac{2\pi}{T}\right)^2.$$

Now substituting this value of $\gamma M/R^3$ into (86·14), we have

$$\left(\frac{d\psi}{dt}\right)_s = -\frac{3}{2\Omega}\left(\frac{2\pi}{T}\right)^2\left(\frac{C-A}{C}\right)\cos\theta. \tag{86·16}$$

We take the year as the unit of time. Then with reference to the fixed stars the angular velocity of the earth's rotation is

$$\Omega = 2\pi \times 266\tfrac{1}{4} \text{ radians per year.*}$$

Taking $\dfrac{C-A}{C} = \dfrac{1}{305.6}$ (Tisserand), $\theta = 23°\,27'$, $T=1$, and substituting these quantities into (86·16), we get

$$\left(\frac{d\psi}{dt}\right)_s = -\frac{3(4\pi^2)\,0.9174}{4\pi \times 366\frac{1}{4} \times 305.6} = -0.00007725 \text{ radian per year.}$$

Dividing this result by 0.000004848, the radian value of 1″, we get

$$\left(\frac{d\psi}{dt}\right)_s = -\frac{0.00007725}{0.000004848} = -15''.9 \text{ per year.} \qquad (86{\cdot}17)$$

Up to this point we have assumed that the sun is the attracting body. The same analytical results hold in the case of the moon. In this case equation (86·14) becomes

$$\left(\frac{d\psi}{dt}\right)_m = -\frac{3\gamma m_2}{2\Omega m r_2^3}\left(\frac{C-A}{C}\right)\cos\theta, \qquad (86{\cdot}18)$$

where m_2 denotes the mass of the moon and r_2 is the distance from the earth to the moon.

To eliminate $\gamma m_2/r_2^3$, we utilize the fact that the earth's attraction on the moon is balanced by the centrifugal force acting on the moon in its orbit. But here the attracting force is not simply $\gamma m m_2/r_2^2$ but is $\{\gamma(m+m_2)\,m_2\}/r_2^2$, as is shown in works where the problem of two bodies is considered.† (The second expression was not used in dealing with the attraction of the sun because the mass of the earth is negligible in comparison with that of the sun.) Hence we have

$$\frac{\gamma(m+m_2)\,m_2}{r_2^2} = m_2 r_2 n_2^2, \qquad (86{\cdot}19)$$

where n_2 denotes the angular velocity of revolution of the moon around the earth. If T_2 denotes the period of the moon's revolution around the earth, the equation for circular motion gives

$$2\pi = n_2 T_2,$$

* The earth actually rotates about its axis $366\frac{1}{4}$ times a year, $365\frac{1}{4}$ times with respect to the sun and once in going around the sun.

† See, for example, Appell's *Traité de Mécanique Rationnelle*, vol. I (1919), pp. 413 and 416.

from which $n_2 = 2\pi/T_2$. Hence (86·19) becomes

$$\frac{\gamma(m+m_2)}{r_2^3} = \left(\frac{2\pi}{T_2}\right)^2. \tag{86·20}$$

Since the mass of the earth is 81·5 times the mass of the moon, we have

$$m = 81.5m_2.$$

Substituting this value of m in (86·20), we have

$$\frac{\gamma(82.5)\,m_2}{r_2^3} = \left(\frac{2\pi}{T_2}\right)^2,$$

or

$$\frac{\gamma m_2}{r_2^3} = \frac{1}{82.5}\left(\frac{2\pi}{T_2}\right)^2.$$

Then substituting this value of $\gamma m_2/r_2^3$ into (86·18), we get

$$\left(\frac{d\psi}{dt}\right)_m = -\frac{3}{2\Omega}\,\frac{1}{82.5}\left(\frac{2\pi}{T_2}\right)^2\left(\frac{C-A}{C}\right)\cos\theta. \tag{86·21}$$

The period of the moon's revolution around the earth is 27.32 days. Hence

$$T_2 = \frac{27.32}{365\frac{1}{4}} \text{ year.*}$$

Using this value for T_2 in (86·21), taking

$$\frac{C-A}{C} = \frac{1}{305.6}$$

and $\theta = 23°\,27'$, we have

$$\left(\frac{d\psi}{dt}\right)_m = -\frac{3}{2\times 2\pi\times 366\frac{1}{4}}\times\frac{1}{82.5}\times 4\pi^2\left(\frac{365\frac{1}{4}}{27{\cdot}32}\right)^2\frac{0{\cdot}9174}{305{\cdot}6}$$

$$= -0.0001674 \text{ radian per year}$$

$$= -34''.5 \text{ per year}$$

Hence the total precession due to both sun and moon is

$$-15''.9 - 34''.5 = -50''.4 \text{ per year.}$$

* Here we use $365\frac{1}{4}$ instead of $366\frac{1}{4}$ because we are concerned with the time required for the earth to revolve around the sun. The required time is $365\frac{1}{4}$ days.

For a precession of 1° the required time is $3600/50.4 = 71.4$ years. The time required for a complete revolution of the vernal equinox around the sun is

$$\frac{360 \times 60 \times 60}{50.4} = 25{,}700 \text{ years.}$$

87. Other astronomical applications

(*a*) *Retrogression of the Moon's Nodes.* The moon's nodes are the points where its orbit intersects the plane of the ecliptic. They correspond to the earth's equinoctial points (equinoxes). They have a retrograde motion and make one revolution in the plane of the ecliptic in 18.6 years.

The retrogression of the moon's nodes can be explained by gyroscopic principles. The interested reader will find the matter treated in Klein and Sommerfeld, Heft 3, in Greenhill's *Report of Gyroscopic Theory*, p. 176, and in A. Gray's *Gyrostatics and Rotational Motion*, pp. 219–25.

(*b*) *Variation of Latitude.* The earth's axis of rotation does not coincide exactly with its geometric axis of figure. The north geographical pole, for example, wanders in a circle of about 50 feet in diameter. This wandering of the pole causes a slight variation in the latitude of any place.

If the earth is assumed to be an absolutely rigid body, the period of the pole variation is found by gyroscopic principles to be

$$\frac{C}{C-A} = 305.6 \text{ days.}$$

But the earth is not an absolutely rigid body. It is slightly elastic, and its elasticity causes the period of the pole's wandering to be about 428 days. The interested reader will find the matter fully treated in Klein and Sommerfeld, in Greenhill's Report, and in Gray's *Gyrostatics and Rotational Motion*, Chapter 11.

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APPENDIX

Schuler Tuning of Gyroscopic Compasses and Gyroscopic Pendulums

1. Introduction

When gyroscopic compasses and gyroscopic pendulums are installed on ships, airplanes and other vehicles, the proper functioning of these gyroscopic instruments is liable to be disturbed by the motions of the vehicles on which they are installed. We saw in Art. 72, for example, that the rolling of a ship will produce a forced vibration in a gyroscopic pendulum suspended on the ship. Such facts and considerations led Max Schuler, of Germany, to a study of the conditions which must be satisfied if the performance of gyroscopic instruments is not to be disturbed by the motions of the vehicles on which the instruments are installed. He found that if the instruments are not to be affected by the motions of the vehicles, the vibration periods of the instruments must be equal to $2\pi\sqrt{(R/g)}$, where R denotes the radius of the earth and g is the acceleration of gravity. This vibration period is about 84 minutes. Schuler tuning consists in so designing gyroscopic instruments that their vibration periods shall be about 84 minutes. Such design is easily attainable for gyroscopic compasses, but is scarcely within the limits of practicability for gyroscopic pendulums.

To get some idea of the principle underlying Schuler tuning, let us consider a gyroscopic pendulum suspended in a vehicle that is free to move in any direction over the earth's surface, and let R denote the earth radius drawn to the moving vehicle, as indicated in Fig. 69. We shall disregard the earth's rotation and consider the motion of a vehicle on a fixed earth.*

* The earth rotates through 15° every hour, and this alone would tend to divert the spin axis of the gyro from the vertical by 15° per hour. But, because the earth's rotation is perfectly uniform, a suitable mechanism can compel the spin axis to rotate at the same rate and thus remain vertical when the point of support is at rest with respect to the earth.

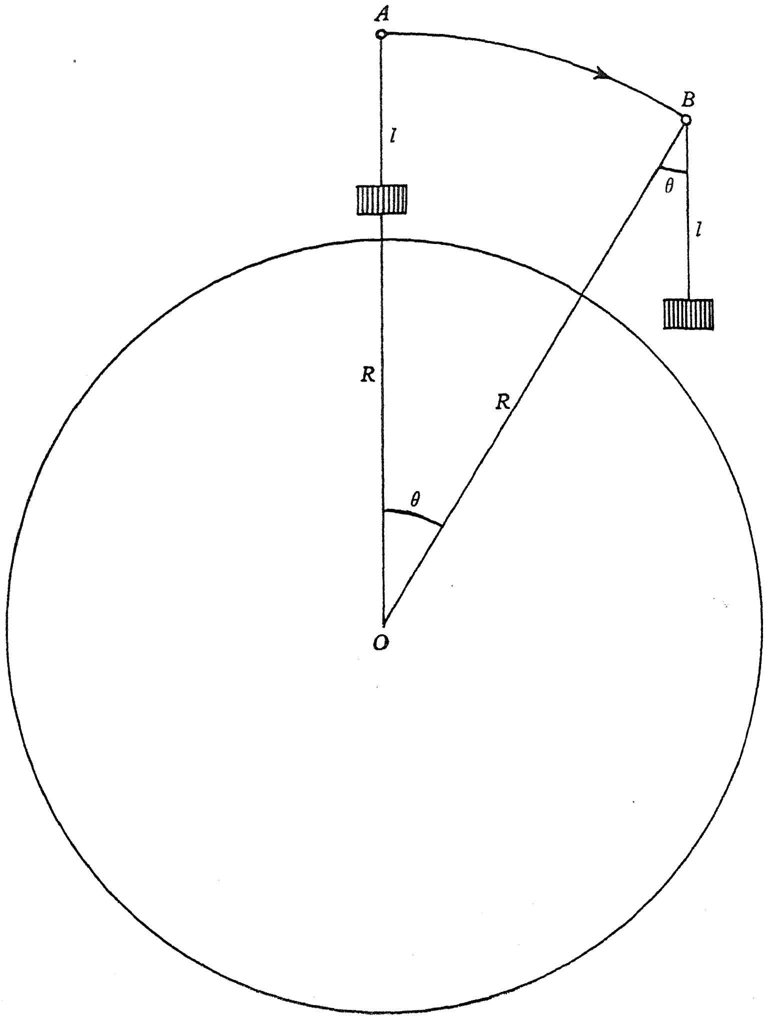

Fig. 69

If the point of suspension moves from A to B with velocity v, the earth radius drawn to the vehicle rotates through the angle θ with angular velocity $d\theta/dt$, or ν say, so that $v = R(d\theta/dt) = R\nu$. Because of the tendency of a spinning gyro to maintain its spin axis in the same direction while the vehicle moves in any manner, the spin axis will not be vertical when the vehicle reaches B, but

will have departed from the vertical by the angle θ. Hence if the spin axis is to be kept vertical while the vehicle is moving, it must be rotated simultaneously with the same angular velocity and angular acceleration as the earth radius drawn to the vehicle. The motion of the pendulum must therefore be a translation combined with rotation about the point of suspension.

2. Schuler Tuning of a Gyroscopic Pendulum

As an aid in explaining the derivation of the condition for Schuler tuning in a gyropendulum, we have drawn in Fig. 70 a set of rectangular coordinate axes with origin at the point of suspension, but no use will be made of the axes except for reference in the explanation. Assume that the vehicle moves along the path AB (Fig. 70) and let

v = velocity of vehicle along AB (this path may be considered an arc of a great circle of the earth),

R = earth radius drawn to the moving vehicle,

ν = angular velocity of earth radius drawn to the moving vehicle,

a = acceleration of vehicle along AB,

m = mass of pendulum,

W = weight of pendulum $= mg$,

l = length of pendulum.

Then the required moment for rotating the spin axis in the xz-plane will be supplied by gravity if the pendulum is tilted away from the vertical by the small angle β in the yz-plane, as indicated in Fig. 70. The moment produced by this tilt is $Wl \sin \beta$ about the x-axis, and it causes the pendulum to precess in the xz-plane with angular velocity ν. Hence by equation (22·3) we have the relation

$$Wl \sin \beta = C\omega\nu \sin (90° - \beta)$$

$$= C\omega\nu \cos \beta,$$

from which

$$\tan \beta = \frac{C\omega\nu}{Wl} = \frac{C\omega v}{WlR}.$$

Since β is small because of the large value of R, we may replace $\tan\beta$ by β and get

$$\beta = \frac{C\omega v}{WlR} \tag{1}$$

as the equation of equilibrium of the pendulum for any speed v of the vehicle. The rate of change of β from the equilibrium position is

$$\left(\frac{d\beta}{dt}\right)_1 = \frac{C\omega}{WlR}\frac{dv}{dt} = \frac{C\omega}{WlR}a \tag{2}$$

and thus varies with the acceleration of the vehicle.

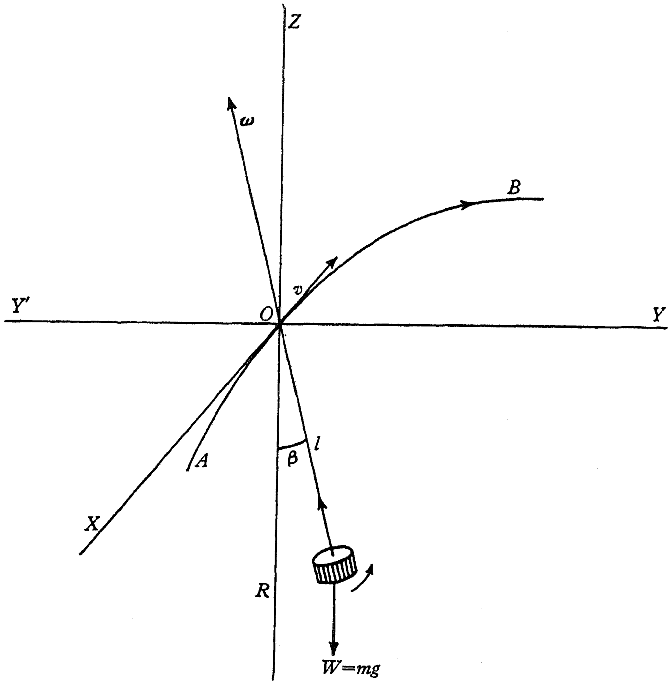

Fig. 70

Now, when the vehicle moves in the xz-plane with acceleration a, it gives the point of suspension the same acceleration. But the pendulum, because of its inertia, resists such acceleration with the inertia reaction force $(W/g)\,a$ acting through the center of gravity of the pendulum. This inertial reaction force exerts a moment $(W/g)\,al$ about the point of suspension and in the xz-plane. This moment, because the vector representing it points in the direction

OY', causes the spin vector ω to precess in the yz-plane with an angular velocity $(d\beta/dt)_2$ toward OY'. Then by equation (22·4) we have the relation

$$\frac{W}{g}al = C\omega\left(\frac{d\beta}{dt}\right)_2. \tag{3}$$

If the pendulum is to remain vertical while the vehicle is moving in any manner, we must have

$$\left(\frac{d\beta}{dt}\right)_2 = \left(\frac{d\beta}{dt}\right)_1.$$

Then on making these substitutions in (2) and (3), we get

$$\frac{C\omega}{WlR}a = \frac{Wla}{gC\omega}$$

or

$$\frac{C\omega}{Wl} = \sqrt{\frac{R}{g}}. \tag{4}$$

But, since $C = mk^2 = (W/g)\,k^2$, where k is the radius of gyration of the pendulum about the spin axis, we get the simple relation

$$\frac{k^2\omega}{l} = \sqrt{(Rg)} \tag{5}$$

as the condition for the Schuler tuning of a gyropendulum.

We found in Art. 69 that the precession period of a gyropendulum is given by the relation $T = 2\pi(C\omega/Wl)$. Hence on replacing $(C\omega/Wl)$ by $\sqrt{(R/g)}$ from (4), we get

$$T = 2\pi\sqrt{\frac{R}{g}}. \tag{6}$$

Taking $R = 20{,}890{,}000$ feet and $g = 32.16$ ft./sec.^{2}, we find from (6)

$$T = 5064 \text{ seconds}$$

$$= 84.4 \text{ minutes.}$$

On substituting in equation (1) the value of $C\omega/Wl$ from (4), we get

$$\beta = \frac{v}{\sqrt{(Rg)}} \tag{7}$$

as the equilibrium angle for a Schuler tuned gyropendulum.

Numerical Example. To get an idea of the magnitude of β as given by (7), assume that $v=60$ miles per hour $=88$ feet per second, $R=20{,}890{,}000$ feet, and $g=32.2$ ft./sec.^{2}. Then from (7) we get

$$\beta=0.003392 \text{ radian}$$

$$=11'\,40''.$$

If $v=300$ miles per hour, we find $\beta=58'\,20''$, or $1°$ approximately. The required tilt is thus very small.

In Art. 69 we found that when a gyropendulum is disturbed while its gyro is spinning rapidly, the pendulum bob will slowly precess in a circular path, the supporting rod describing a conical surface. The results in the above example show that in a Schuler tuned pendulum the radius of the circular path is exceedingly small, being $l\beta=0.00339l$ for a vehicle speed of 60 miles per hour.

In order to examine the practicability of Schuler tuning for gyropendulums, let us consider a numerical case. Assume that the rotor of a gyropendulum is 6 inches in diameter, that the rim is 2 inches wide and 1 inch thick, and that the web of the rotor is $\frac{1}{2}$ inch thick. Then if δ denotes the density of steel, we find

$$\text{moment of inertia of rotor}=\frac{23\pi\delta}{96\times 864},$$

$$\text{mass of rotor}=\frac{\pi\delta}{144},$$

$$k^2=\frac{23\pi\delta}{96\times 864}\div\frac{\pi\delta}{144}=\frac{23}{576}.$$

If

$$\omega=12{,}000 \text{ r.p.m.}=400\pi \text{ radians per second,}$$

$$R=20{,}890{,}000 \text{ feet,}$$

$$g=32.16 \text{ ft./sec.}^2,$$

we find from (5) $\qquad l=0.000308\times 2\pi$ feet

$$=0.023 \text{ inch.}$$

This short distance between the point of suspension and the center of gravity of the pendulum makes the construction of such a pendulum a rather difficult matter.

3. Schuler Tuning of the Gyroscopic Compass

In Art. 56 we found that when a ship is sailing in a northerly direction with speed v and on a course making an angle θ with the north meridian, the gyrocompass points slightly to the west of north at an angle γ given by the equation

$$\gamma=\frac{v\cos\theta}{R\Omega\cos\lambda}=\frac{v_n}{R\Omega\cos\lambda}, \tag{1}$$

where $v_n(=v\cos\theta)$ denotes the northern component of the velocity. This equation gives the equilibrium position of the gyro axis for any speed and direction of the ship. The rate of change of γ from the equilibrium position is

$$\left(\frac{d\gamma}{dt}\right)_1=\frac{1}{R\Omega\cos\lambda}\,\frac{dv_n}{dt}=\frac{a_n}{R\Omega\cos\lambda}, \tag{2}$$

where a_n denotes the northern acceleration of the point of suspension of the compass and γ increases westward.

When the point of suspension is given a northward acceleration, the inertia of the pendulous weight w resists such acceleration by the inertia force $(w/g)\,a_n$, and this exerts a moment $r(w/g)\,a_n$ about the point of suspension and in the plane of the meridian, where r is the moment arm of w. Since the vector representing this moment points toward the west, the gyro axis precesses toward the west at angular velocity $(d\gamma/dt)_2$ in the plane elevated at the angle β above the horizon (Fig. 46). Hence by equation (24·4) we have

$$r\frac{w}{g}a_n=C\omega\left(\frac{d\gamma}{dt}\right)_2. \tag{3}$$

Then, if there is to be no disturbance in the indication of the compass, we must have

$$\left(\frac{d\gamma}{dt}\right)_2=\left(\frac{d\gamma}{dt}\right)_1.$$

Equating the values of these derivatives from (2) and (3), we get

$$\frac{a_n}{R\Omega\cos\lambda}=\frac{wra_n}{gC\omega},$$

from which

$$\frac{C\omega}{wr}=\Omega\left(\frac{R}{g}\right)\cos\lambda. \tag{4}$$

This equation expresses the condition for Schuler tuning of a gyrocompass.

In Art. 52 we found that the long vibration period of a gyrocompass is given by the formula

$$T = 2\pi\sqrt{\frac{C\omega}{wr\Omega\cos\lambda}}.$$

On substituting the value of $wr\Omega\cos\lambda$ from (4), we get

$$T = 2\pi\sqrt{\frac{R}{g}} \tag{5}$$

which is the same period as found for the gyropendulum and therefore gives a period of 84.4 minutes.

Fortunately, it is possible and practicable to construct gyroscopic compasses that will satisfy condition (4), and modern gyrocompasses are so constructed. Gyrocompasses therefore indicate true north regardless of the motions of the ships on which they are installed.

When equation (47·1) is combined with (4) above, the result is

$$\sin\beta = \frac{R\Omega^2 \sin 2\lambda}{2g},$$

or, since β is very small,

$$\beta = \frac{R\Omega^2 \sin 2\lambda}{2g} \tag{6}$$

which gives the elevation of the north end of the gyro axis when in its equilibrium position. For $\lambda = 45°$ we get

$$\beta = 0.00173 \text{ radian}$$

$$= 6' = 0°.1, \quad \text{approximately.}$$

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